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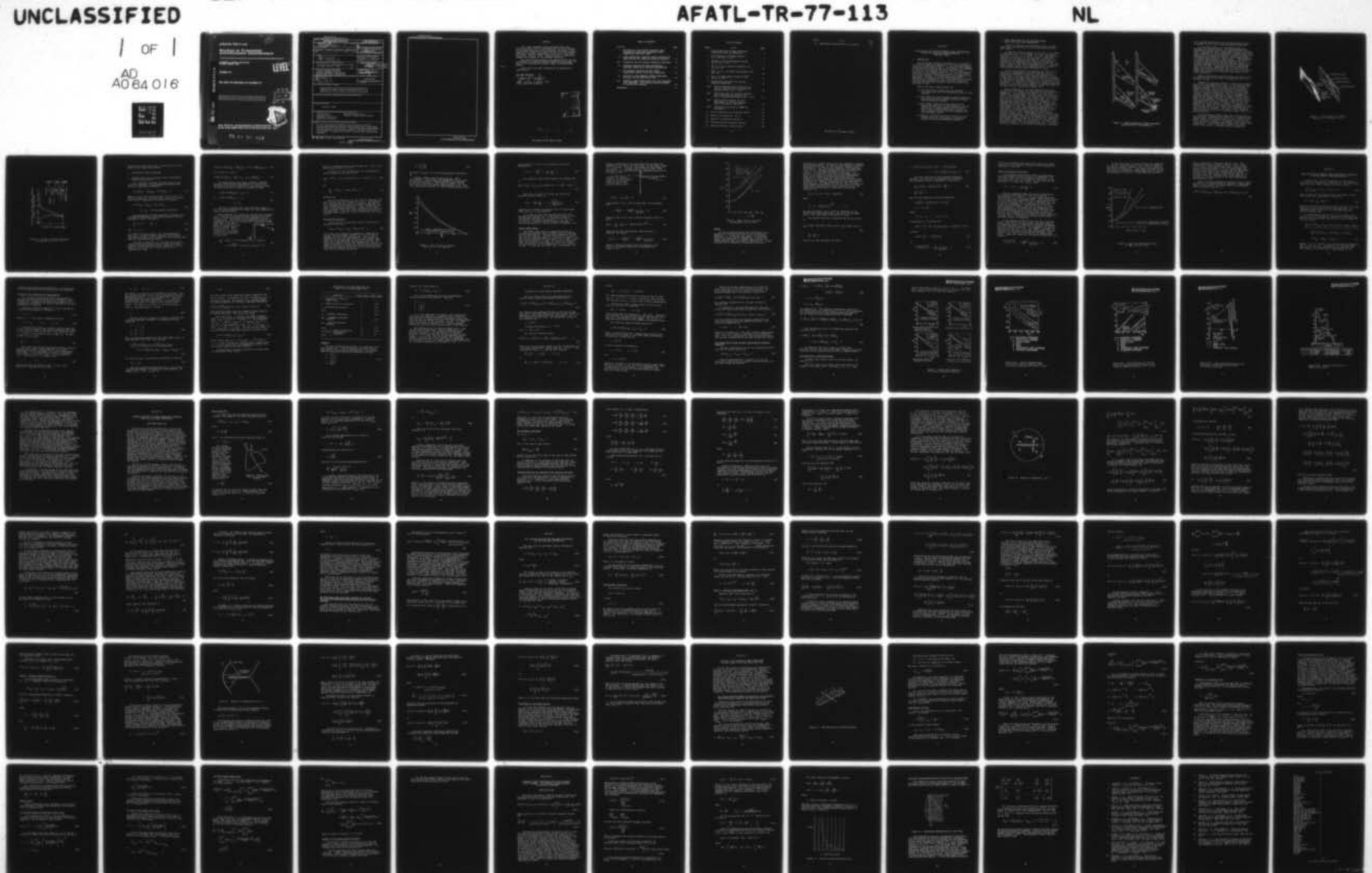

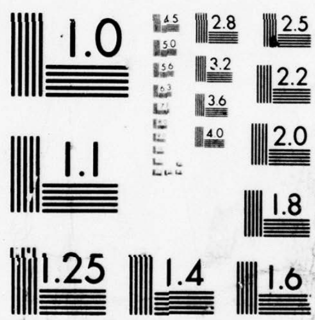

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Studies in Transonic Aerodynamic Interference

THE UNIVERSITY OF TENNESSEE SPACE INSTITUTE
TULLAHOMA, TENNESSEE 37388

SEPTEMBER 1977

FINAL REPORT FOR PERIOD MARCH 1977-SEPTEMBER 1977

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Air Force Armament Laboratory

AIR FORCE SYSTEMS COMMAND ★ UNITED STATES AIR FORCE ★ EGLIN AIR FORCE BASE, FLORIDA

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER (18) AFATL (19) TR-77-113	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER (9)
4. TITLE (and Subtitle) (6) STUDIES IN TRANSONIC AERODYNAMIC INTERFERENCE.		5. ~~TYPE OF REPORT & PERIOD COVERED~~ Final Report, 1 March 1977 - September 1977
		6. ~~PERFORMING ORG. REPORT NUMBER~~
7. AUTHOR(s) (10) S. N. Chaudhuri		8. CONTRACT OR GRANT NUMBER(s) (15) F08635-77-C-0098
9. PERFORMING ORGANIZATION NAME AND ADDRESS The University of Tennessee Space Institute Tullahoma, Tennessee 37388		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS JON (16) 2307 E301 (17) E3 PE: 61102F
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Armament Laboratory Armament Development & Test Center Eglin Air Force Base, Florida 32542		12. ~~REPORT DATE~~ (11) September 1977
		13. ~~NUMBER OF PAGES~~ 88
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 95p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Available in DDC.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Transonic Flow
Compressible Flow
Aerodynamic Interference
Wing-Store Interference
Wing-Body Transonic Flow
Transonic Integral Equation Methods

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The work reported herein represents the first phase of a study of transonic aerodynamic interference problems as applicable to store-wing-body combinations without the pylon. The nonlifting and lifting transonic integral equations in the velocity components in the field of the flow were obtained. Two methods of solution of the nonlifting case have been described.

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PREFACE

This report documents work performed during the period from March 1977 to September 1977 by The University of Tennessee Space Institute, Tullahoma, Tennessee 37388, under Contract F08635-77-C-0098 with the Air Force Armament Laboratory, Armament Development & Test Center, Eglin Air Force Base, Florida 32542. Captain Robert A. Grow (DLJC) managed the program for the Armament Laboratory.

This report has been reviewed by the Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS it will be available to the general public, including foreign nations.

This report has been reviewed and is approved for publication.

FOR THE COMMANDER

John R Taylor

JOHN R. TAYLOR, Lt Colonel, USAF
Chief, Munitions Division

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SECTION I

DISCUSSION OF THE VARIOUS TRANSONIC SMALL PERTURBATION EQUATIONS DESCRIBING THREE-DIMENSIONAL WING-BODY FLOWS

1. INTRODUCTION

One of the most difficult problems yet to be solved to a certain degree of completion is the three-dimensional transonic interference problem. At transonic Mach numbers, three-dimensional effects due to the presence of a fuselage or wing tip can influence the flow field for large distances in the spanwise direction. For low and moderate aspect-ratio wings combined with a body, the flow fields are rarely similar to two-dimensional ones anywhere on the wing except, perhaps, on certain localized regions. Therefore, the treatment of the three-dimensional transonic flow problems involves more than a trivial extension of existing two-dimensional methods, which are fairly well developed by now.

Some of the major complications are:

(1) The circulation changes with the spanwise coordinate and has to be determined as part of the solution.

(2) The swept and tapered planform shapes complicate application of wing boundary conditions using finite-difference methods.

(3) The shocks, which are nearly normal to the free-stream direction in two dimensions, can be oblique in the lateral direction for finite swept wings, thereby making them difficult to capture sharply because of poor alignment with the coordinate system.

(4) Finally, difficulties in the stability of the results may arise, especially in supersonic regions in the finite-difference method.

2. BRIEF DESCRIPTION OF THE FLOW DEVELOPMENT ABOUT SWEPT WING-BODY COMBINATION

Figure 1 shows the flow development about a simple swept wing-body combination with increasing free-stream Mach number.

At moderate subsonic speeds, if the lift is not large, an important effect of the sweep is to move the loading forward at the tip. As a result, the first appearance of supersonic flow occurs in the tip region. Since the isobars near the wing tip lose much of their sweep, the supersonic flow generally gives rise to a shock wave--the initial tip shock which is comparatively weak and lies almost normal to the free stream [Figure 1(a)]. This tip shock, though limited in the spanwise extent, extends to considerable distances above and below the wing.

As the free-stream M_∞ is increased, the initial tip shock moves rearwards over the wing surface, but its influence on the wing is limited by the appearance of a second shock. This rear shock rapidly develops to affect a large part of the wing span, particularly the flow ahead of the initial tip shock [Figure 1(b)]. The rear shock may be regarded as associated primarily with conditions at the wing root. It consists of a compression system propagating outward from the root which coalesces on the outer part of the wing to form a shock wave [Figure 1(b)].

With increasing M_∞, the rear shock moves aft more rapidly than the initial tip shock which is overtaken and disappears. At a sufficiently high M_∞ the rear shock reaches the trailing edge. The high local flow velocities close to the leading edge over the outer part of the wing lead to flow separation at the leading edge. This starts near the tip and spreads inboard with increasing α. For leading-edge sweeps greater than about 30 degrees, the separated flow rolls up to form a partspan vortex lying obliquely across the wing. Above a certain stream M_∞ the flow changes in type; the leading-edge separation is suppressed, the flow passes smoothly around the leading edge through a forward shock wave which appears to originate close to the leading edge at some spanwise position [Figure 1(c)]. The forward shock is in fact the boundary of disturbances from the inboard part of the wing leading edge which, because of the higher local supersonic Mach number, propagate over the outer wing in a direction more highly swept than the leading edge. The Mach number component normal to the leading edge ($M_\infty \cos \Lambda$) for which the flow attachment occurs varies considerably, depending

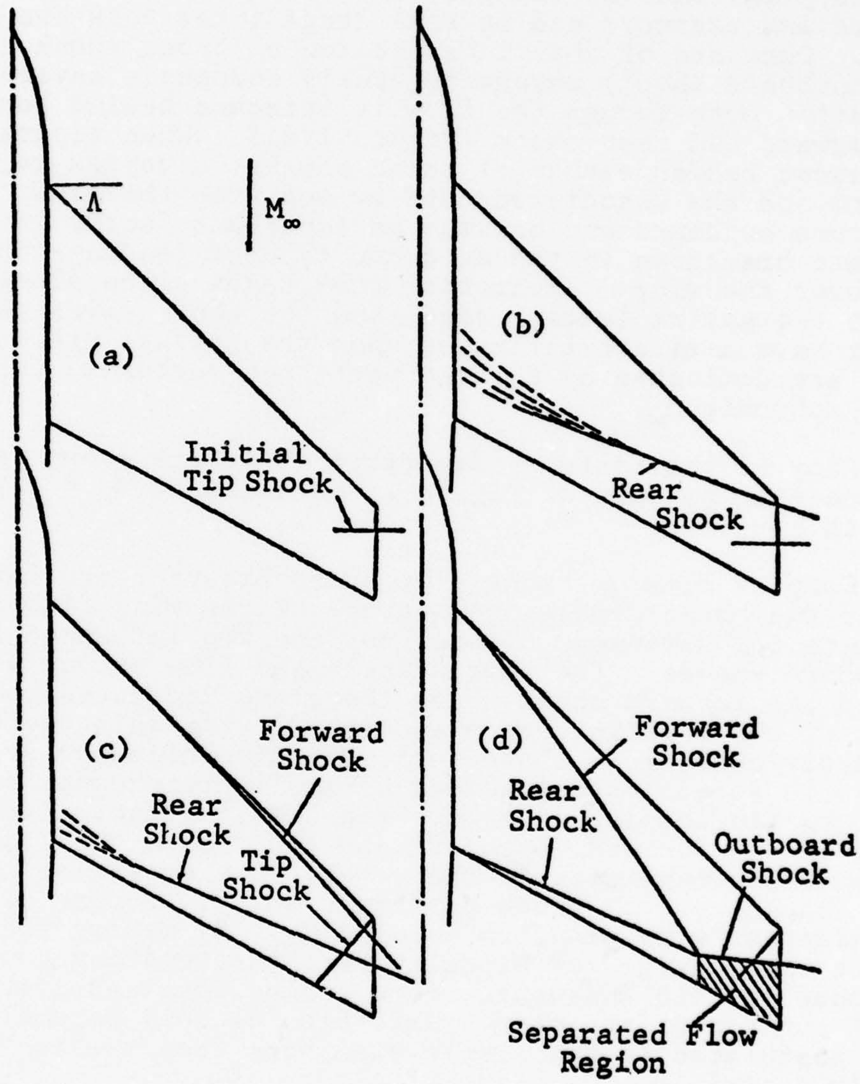


Figure 1. Shock Formations on Swept Wing-Body Combination with Increasing M_∞.

on Λ, and more particularly on the leading edge profile. In general $M_\infty \cos \Lambda$ has a value between 0.55 and 0.90.

As the free-stream M_∞ is increased beyond that necessary for flow attachment, the forward shock moves inboard and rearward and at some stage intersects the rear shock. Outboard of this intersection a strong shock forms (the outboard shock) which frequently induces a severe flow separation even though the flow is attached behind both the forward and rear shock [Figure 1(d)]. When separation is present behind either of these shocks, a vortex tends to form and the associated outflow modifies the flow structure outboard and becomes an important factor in the ultimate breakdown in the attached type of leading-edge flow over the wing. Separation then takes place along almost the entire leading edge, and the shock waves no longer have a direct influence upon the surface pressures which are dominated by a large port-span vortex as for lower subsonic M_∞.

Many of the features described above are shown in the surface-film pattern of Figure 2, obtained at $M_\infty = 0.95$ or with $\alpha = 10^o$.

Complex flow patterns similar to Figure 2 are largely due to the three-dimensional nature of the wing flow and hence to the dominance of the root and tip influence at transonic speeds. The root affects the flow strongly behind the forward shock which therefore indicates the limit of the root influence and the flow in this region is partly conical in character. The tip influence at transonic speeds is delineated by small disturbance from near the tip leading edge [the tip shock: Figure 1(c)]. Ahead of the forward, outboard and tip shocks the flow is almost two-dimensional in character. The extent of this zone depends on the shock positions which, in turn, are dependent on wing planform, α, and M_∞. By making the aspect ratio large, or by deliberately attempting to reduce the root and tip influence, flow can be obtained over a large portion of the wing. This flow closely resembles that postulated in the simple sweepback theory which is so highly desirable.

The relationship between swept-wing flow and that on the equivalent two-dimensional section at a component Mach number normal to the leading edge, $M_\infty \cos \Lambda$ (if not too large) is shown in Figure 3. At higher values of $M_\infty \cos \Lambda$ it is difficult to minimize the effects of the finite wing aspect-ratio. Recent experiments however have given satisfactory results: surface pressures, separation

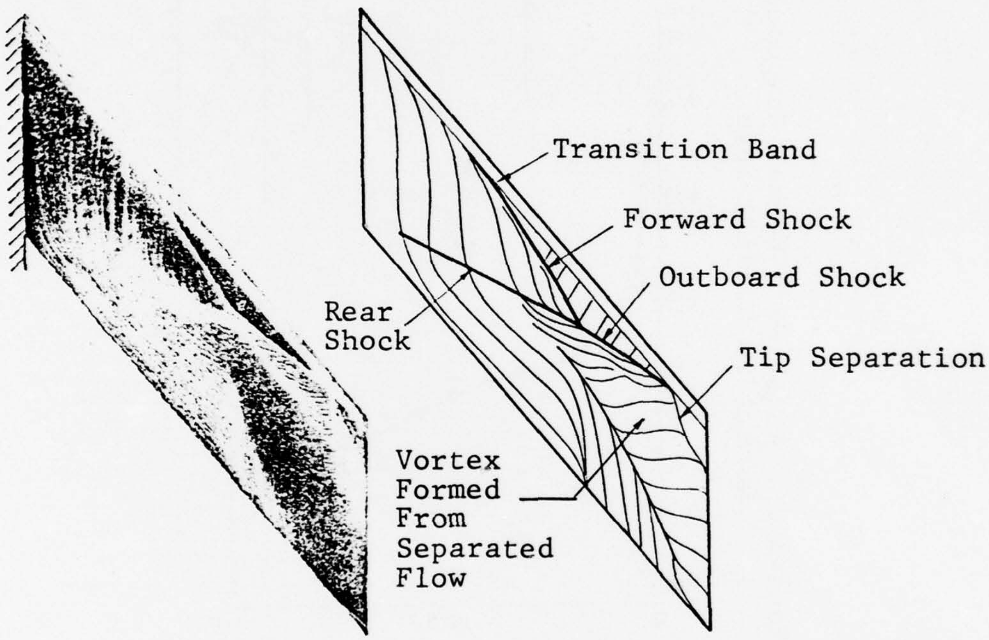


Figure 2. Flow Pattern on 50^{o} Swept Wing at $M_{\infty} = 0.95$ and $\alpha = 10^{o}$.

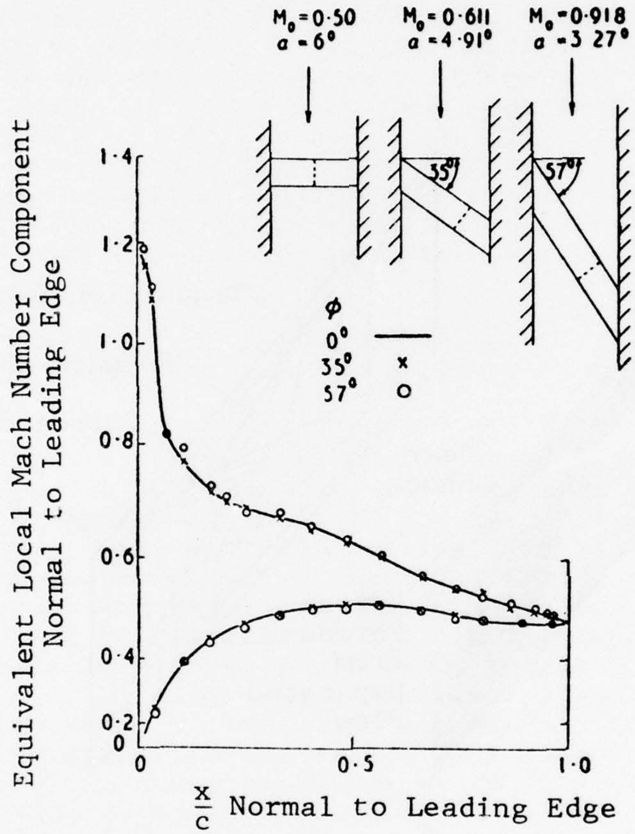


Figure 3. Validity of Simple Sweepback Concept for $M_\infty \cos \Lambda = 0.5$.

boundaries and shock positions correlating well on the basis of the simple sweep theory.

3. DISCUSSION OF BASIC EQUATIONS

A. Various Forms of the Transonic Small Perturbation Differential Equations

First consider the exact nonlinear equation for inviscid, isentropic, two-dimensional potential flow written in Cartesian coordinates:

$$(a^2-\Phi_x^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (a^2-\Phi_y^2)\Phi_{yy} = 0 \tag{1}$$

where a is the local speed of sound, and Φ, the total velocity potential. In terms of $U = \Phi_x$ and $V = \Phi_y$, the total velocity components of Equation (1) become

$$\begin{gathered}(a^2-U^2)U_x = 2UV\,U_y - (a^2 - V^2)V_y \quad , \\ V_x = U_y \quad \text{(irrotationality condition)}\end{gathered} \tag{2}$$

The expressions of these equations in terms of the perturbation potential ϕ and the perturbation velocity components u and v, defined by

$$\frac{U}{U_\infty} = 1 + u \quad ; \quad \frac{V}{V_\infty} = v \tag{3}$$

and $$\Phi = x + \phi \tag{4}$$

can easily be written down in full (see Reference 1, page 204). We will, however, consider two of these perturbation forms for our guidance in selecting the best form for three-dimensional flows.

One form is referred to as the small perturbation equation derived by neglecting all terms containing products of perturbation velocity components. We will refer to this as *NASA Ames (transonic) equation* (Reference 2) (NAE)

$$[1-M_\infty^2-(\gamma+1)M_\infty^2\phi_x]\phi_{xx} - 2M_\infty^2\phi_y\phi_{xy} + [1-(\gamma-1)M_\infty^2\phi_x]\phi_{yy} = 0 \tag{5}$$

or in terms of u and v:

$$[1-M_\infty^2-(\gamma+1)M_\infty^2 u]u_x = 2M_\infty^2\ v\ u_y - [1-(\gamma-1)M_\infty^2 u]v_y \tag{6}$$

The second form is the usual classical transonic equation (CTE) derived from Equations (5) and (6) by considering the special behavior of the fluid flow when the local velocity is close to the speed of sound:

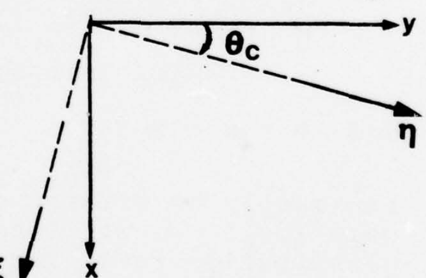
$$[1-M_\infty^2-(\gamma+1)M_\infty^2\phi_x]\phi_{xx} + \phi_{yy} = 0 \tag{7}$$

or

$$\left.\begin{aligned}[1-M_\infty^2-(\gamma+1)M_\infty^2 u]u_x &= - v_y\\ v_x &= u_y\end{aligned}\right\} \tag{8}$$

We will now analyze these equations with regard to how they model characteristic lines, sub- and supercritical flows, and shock waves.

When Equations (1) through (8) are used for supersonic flows, they can be analyzed to find characteristic lines; that is, lines across which velocity gradients can be discontinuous. One simple way to derive the equation for a characteristic line is to rotate the coordinates through some angle θ_c and ask if there is a value of θ_c for which an advance along ξ is not possible. For Equation (2), it is easy to show that a characteristic line occurs in its solution when



$$\left[\frac{1}{M_\infty^2} - \cos^2\theta_c\right] + [2v\cos\theta_c\sin\theta_c - u(\gamma-1+2\cos^2\theta_c] - [(\gamma-1)\left(\frac{u^2+v^2}{2}\right) + (u\cos\theta_c - v\sin\theta_c)^2] = 0 \tag{9}$$

where the brackets enclose terms having zero-, first- and second-order powers of u and v.

Similarly, it can be shown that the characteristic lines of Equations (6) and (8) are

$$\left[\frac{1}{M_\infty^2} - \cos^2\theta_c\right] + \left[2v\cos\theta_c\sin\theta_c - u(\gamma - 1 + 2\cos^2\theta_c)\right] = 0 \qquad (10)$$

and

$$\left[\frac{1}{M_\infty^2} - \cos^2\theta_c\right] - u(\gamma+1)\cos^2\theta_c = 0 \qquad (11)$$

respectively.

Notice that Equations (9) and (10) are identical for both the zero and first powers of u and v. But Equation (11), derived from the classical transonic equation, is not in agreement with Equation (9) even through the first order in perturbation velocities. Since, to the first-order, oblique shock waves bisect characteristic lines of the same family, this result can influence our choice of equations when we seek to *compute flow fields with oblique shocks*.

The Critical Velocity

If we express a second-order partial differential equation in the form

$$A\,\Phi_{xx} + B\,\Phi_{xy} + C\,\Phi_{yy} + f(\Phi_x, \Phi_y, \Phi) = 0 \qquad (12)$$

it is said to be elliptic or hyperbolic, depending on whether the discriminant (B^2-4Ac) is <0 or >0, and it changes type as it goes through zero. This occurs in Equations (1) and (2) when $a^2(Q^2-a^2)$ goes through zero, where $|Q|$ is the magnitude of the local velocity. This gives the classical result that Equations (1) and (2) change type which the local flow speed passes through the speed of sound and is hyperbolic on the supersonic side. This speed is referred to as the *critical speed* and the value of the pressure coefficient

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} \tag{13}$$

at which it occurs, as the critical pressure coefficient, C_p^*.

Figure 4 shows a plot of C_p^* *versus* M_∞. This relationship is valid under the assumption that the flow is isentropic along a streamline between the reference M_∞ and any given point. Also shown in Figure 4 is the first-order small perturbation approximation to $C_p^* \approx 2u^*$. This approximation follows at once by neglecting all terms

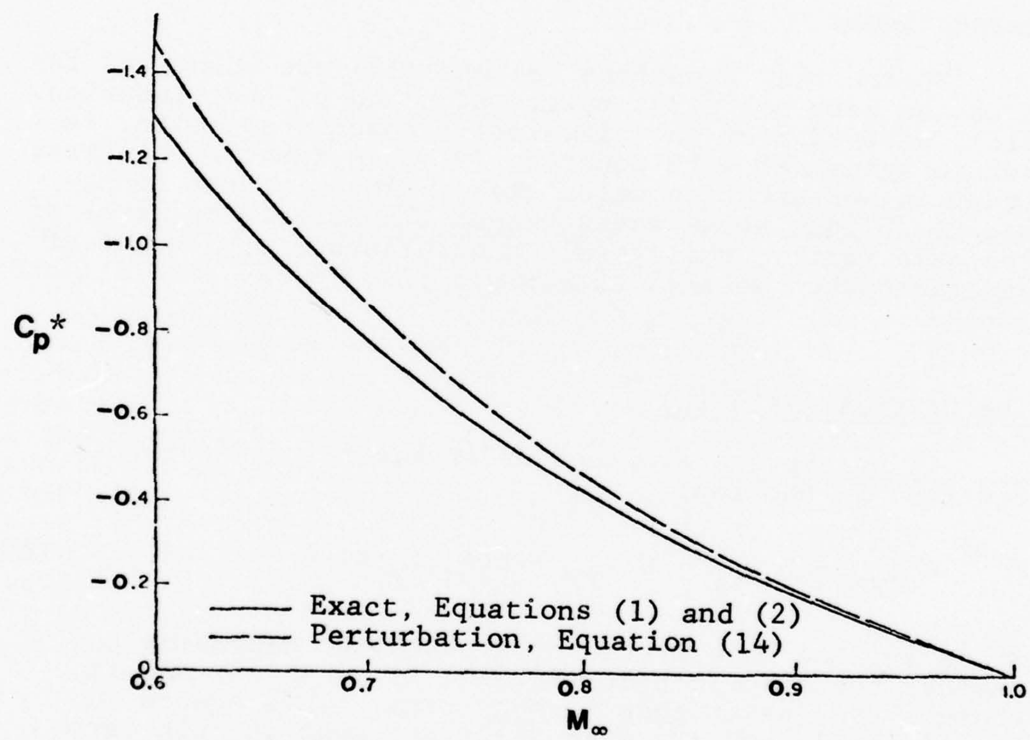


Figure 4. C_p^* *versus* M_∞ as Given by Equations (1), (2) and (14).

with products of u and v in C_p-equation and can be expressed as

$$- 2u^* \simeq 1 - \left(\frac{q^*}{q_\infty}\right)^2 = - \frac{2}{\gamma+1} \left(\frac{1}{M_\infty^2} - 1\right) \tag{14}$$

The condition under which Equation (5) changes type is:

$$M_\infty^4(\gamma^2-1)\phi_x^2 - M_\infty^2 \phi_x [\gamma+1+(1-M_\infty^2)(\gamma-1)] + (1-M_\infty^2) - M_\infty^4 \phi_y^2 = 0 \tag{15}$$

There are two roots for $M_\infty^2 \phi_x^*$, and the one of physical interest leads to

$$- 2u^* = - \frac{2}{\gamma+1} \left[\frac{1}{M_\infty^2} - 1 - \frac{M_\infty^2 \phi_y^2}{1 - \frac{\gamma-1}{\gamma+1}(1-M_\infty^2)}\right] \tag{16}$$

Equation (16) reduces to Equation (14) if ϕ_y^2 and higher order terms in ϕ_y are neglected.

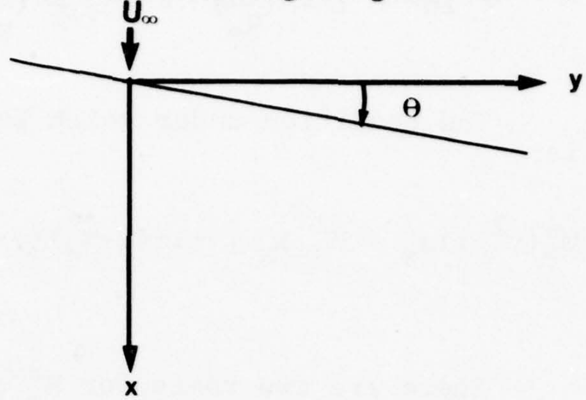
It is easy to show that the expression for -2u* derived from Equation (7) is identical to Equation (14). We therefore conclude that, to the lowest order in perturbation velocities, Equations (1), (2), (5), (6), (7) and (8) all change type according to the same relationship among u, γ and M_∞ as given by Equation (14).

Simple Sweep Theory

The simple sweep theory is generally derived by considering a sheared wing of infinite span and constant section (Reference 3). The principal assumption is that the velocity component parallel to an edge is constant or that the perturbation velocity parallel to an edge is zero. Under these conditions a shock, if it occurs, would also have to be parallel to an edge, and hence its strength would depend only on the component of velocity normal to an edge. We will use the symbol $C_p^*(\theta)$ to designate the

pressure coefficient in an isentropic flow at which the component of the total velocity normal to some plane, fixed by θ = constant, is sonic. This plane will be assumed to be vertical, *i.e.*, perpendicular to the xy-plane. Note that $C_p^*(0) = C_p^*$, which has the same meaning as given in the previous section.

If then $q(\theta)$ is the component of velocity normal to an isobar plane, classical sweep theory leads at once to the critical value of $q(\theta)$:



$$q^{*2}(\theta) - U_\infty^2 \sin^2\theta = a^2 \tag{17}$$

From Equation (14) it can be shown that, for isentropic flow

$$1 - (\frac{q^*(\theta)}{U_\infty} = - \frac{2\cos^2\theta}{\gamma+1} \left(\frac{1}{M_\infty^2 \cos^2\theta} - 1\right) \tag{18}$$

Equation (18) can be used to derive the exact value for $C_p^*(\theta)$:

$$C_p^*(\theta) = \frac{2}{\gamma M_\infty^2} \left[\left\{\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} M_\infty^2 \cos^2\theta\right)\right\}^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{19}$$

Applying the small perturbation approximation to Equation (19) yields

$$- 2u^*(\theta) \simeq 1 - \left[\frac{q^*(\theta)}{q_\infty}\right]^2 \simeq - \frac{2\cos^2\theta}{\gamma+1} \left(\frac{1}{M_\infty^2 \cos^2\theta}\right) \tag{20}$$

Figure 5 compares Equation (19) with Equation (20), which is the simple sweep-theory counterpart of Equation (14).

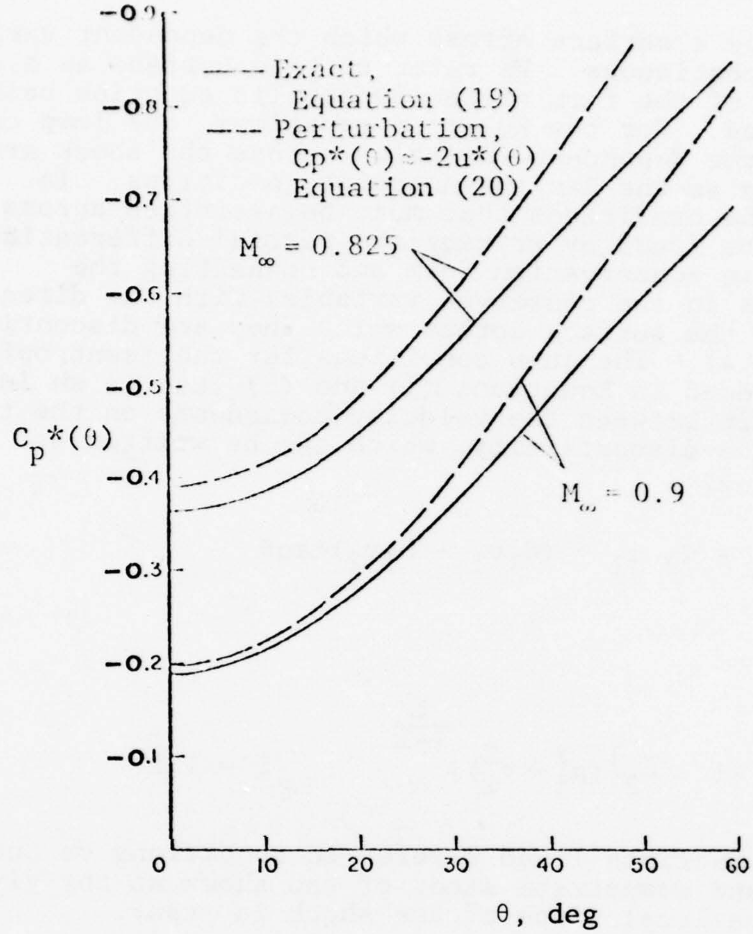


Figure 5. $C_p^*(\theta)$ *versus* θ as Given by Equations (19) and (20).

Shocks

So far the discussion has been restricted to the properties of genuine solutions of the differential equation, *i.e.*, solutions that result in continuous variations in the velocity components. We must next inspect the properties of possible weak solutions of these equations (see, for example, Reference 4). A weak solution can be composed of two genuine solutions

separated by a surface across which the dependent variables can be discontinuous. We refer to this surface as a shock, regardless of the form of the hyperbolic equation being investigated. For the Eulerian equations, the jump conditions for the dependent variables across the shock are referred to as the Rankine-Hugoniot conditions. In general, the conditions that must be satisfied across a shock can be found by writing the partial differential equations in conservation form and connecting the differences in the conserved variables with the direction cosines of the surface across which they are discontinuous (Reference 4). The jump conditions for the isentropic shock embedded in Equations (1) and (2) lead to an implicit relationship between the velocity components on the two sides of the discontinuity, which can be written as (Reference 5):

$$G_1 u_1 - G_2 u_2 = (G_1 v_1 - G_2 v_2)\tan\theta \tag{21}$$

where

$$G_i = \left[1 - \frac{\gamma-1}{2}(u_i^2 + v_i^2)\right]^{\frac{1}{\gamma-1}} \quad ; \quad i = 1,2$$

and the subscripts 1 and 2 refer to conditions on the upstream and downstream sides of the shock at any given point. The local slope of the shock is $-\tan\theta$.

The conservative form of Equation (8) can be written as

$$\frac{\partial}{\partial x}\left[(1-M_\infty^2)u - \frac{3-\gamma}{2} M_\infty^2 v^2 - \frac{1}{2} M_\infty^2 (\gamma+1)u^2\right] + \frac{\partial}{\partial y}\left[v - M_\infty^2(\gamma-1)uv\right] = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{22}$$

and for its jump condition, we obtain

$$[1 - \frac{1}{2} M_\infty^2 (\gamma+1)(u_1+u_2)] \tan^2\theta - 2 M^2(v_2+u_2\tan\theta)$$

$$+ 1 - M_\infty^2 - \frac{1}{2} M_\infty^2(\gamma+1)(u_1+u_2) = 0 \qquad (23)$$

The jump condition for the classical transonic equation, [(7) and (8)] is important for our purpose. The conservative form of Equations (7) and (8) is

$$\frac{\partial}{\partial x} [(1-M_\infty^2)u - \frac{1}{2} M_\infty^2(\gamma+1)u^2] + \frac{\partial v}{\partial y} = 0 \qquad (24)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

and the jump conditions yield the equations:

$$(1-M_\infty^2)\Delta u - \frac{1}{2} M_\infty^2(\gamma+1)\Delta u^2 = \Delta v \tan\theta$$

$$\Delta v = - \Delta u \tan\theta$$

where

$$\Delta u = u_1 - u_2 \quad ; \quad \Delta v = v_1 - v_2 \quad .$$

Eliminating Δv, by noting that

$$\Delta(u^2) = u_1^2 - u_2^2 = (u_1-u_2)(u_1+u_2) = (u_1+u_2)\Delta u = 2\bar{u} \Delta u$$

we have

$$\tan^2\theta = M_\infty^2 - 1 + M_\infty^2(\gamma+1)\bar{u} \qquad (25)$$

or

$$\frac{(-2u_1)+(-2u_2)}{2} \simeq - \frac{2}{\gamma+1} (\frac{1}{M_\infty^2 \cos^2\theta} - 1) \qquad (26)$$

This is the condition that must be met across the shock according to the classical transonic theory expressed by Equations (7) and (8).

Shocks and Supercritical Flow

A flow becomes supercritical when the differential equation changes type from elliptic to hyperbolic. To the lowest order in the perturbation velocities, it has been shown that this occurs under the same condition for Equations (1) through (8), namely, where

$$- 2u = - 2u^* \simeq - \frac{2}{\gamma+1} \left(\frac{1}{M_\infty^2} - 1\right) \tag{27}$$

When all simulated shocks are nearly perpendicular to the free-stream direction, that is to say the approximation $\theta = 0$ and $v_2 = 0$ is good enough, Equations (22) and (24) lead to the same result. This result applies to the standard two-dimensional flow simulations and, in the form of Equation (26), it gives a condition that must be met by the average of the perturbation velocities across the discontinuity. Notice that as $u_1 \rightarrow u_2$, they both approach u*. This means that the intensity of a shock perpendicular to the free stream is centered about the critical perturbation velocity, which, in turn, is based on the condition that the equation changes type. For such cases, Equations (5) through (8) give essentially the same results, and Equations (7) and (8) are to be preferred because of simplicity.

If the boundary conditions are such that a vertical shock is oblique to the free stream, the shock position and intensity could be significantly different depending on whether Equations (5) and (6) or Equations (7) and (8) were used for the simulation. For example, consider the special case where $v_2 = - 2u_2 \tan\theta$, which exists when the component of the perturbation velocity parallel to the shock is zero, which is the condition for simple sweep theory. Under this condition, the relation [Equation (23)] reduces to

$$\frac{(-2u_1)+(-2u_2)}{2} \simeq - \frac{2 \cos^2\theta}{\gamma+1} \left(\frac{1}{M_\infty^2 \cos^2\theta} - 1\right) \tag{28}$$

On the other hand, the jump condition for Equations (7) and (8), given by Equation (26), does not depend on any special relation between u_2 and v_2 and is therefore valid for all values of θ. This simple treatment brings out clearly the difference between Equations (5) and (6) and Equations (7) and (8) in simulating oblique shocks

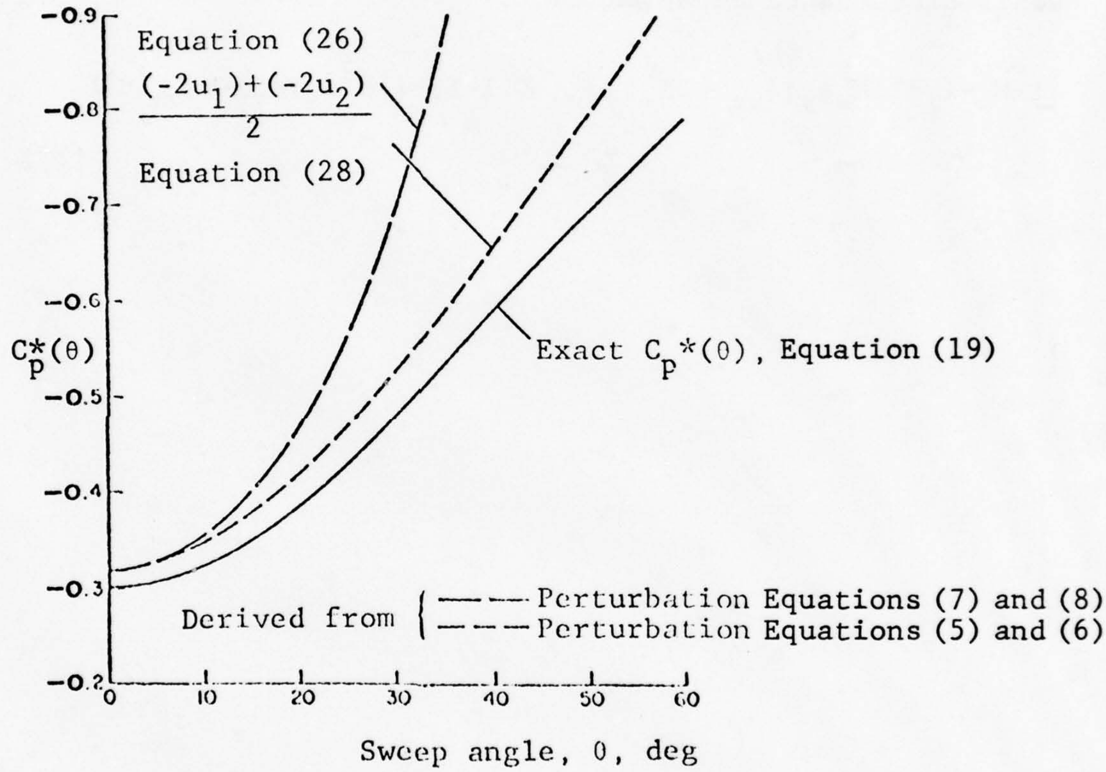


Figure 6. Exact and Approximate Values of $C_p^*(\theta)$ for $M_\infty = 0.85$.

from a comparison of Equations (28) and (26). This difference is shown in Figure 6 for $M_\infty = 0.85$. Exact value of $C_p^*(\theta)$ from simple sweep theory, Equation (19), is also plotted in Figure 6. It is clearly seen from Figure 6 that Equations (7) and (8) are very poor models for flows with shocks that are more than 30 degrees oblique to the free stream. Equations (5) and (6), on the other hand, are acceptable for a wide range of θ, limited only by the accuracy of the small perturbation approximation itself.

Hence, for three-dimensional transonic flows in which the presence of oblique shocks at moderate to large sweep angles is anticipated, the proper form of the transonic small disturbance equation is

$$[1-M_\infty^2-(\gamma+1)M_\infty^2\phi_x]\phi_{xx} - 2M_\infty^2\phi_y\phi_{xy} + [1-(\gamma-1)M^2\phi_x]\phi_{yy} + \phi_{zz} = 0 \tag{29}$$

SECTION II

THREE-DIMENSIONAL TRANSONIC SMALL-PERTURBATION EQUATIONS AND THE ORDER OF MAGNITUDE ANALYSIS

Consider first the exact equation for the velocity potential Φ in rectangular coordinates x', the local stream direction and y' and z' normal to the free stream:

$$\underline{(a^2-q^2)\Phi_{x'x'}} + a^2\Phi_{y'y'} + a^2\Phi_{z'z'} = 0 \tag{30}$$

where a and q are the local values of the speed of sound and stream speed, respectively, related by

$$a^2 = \frac{\gamma-1}{2} + \frac{1}{M_\infty} - \frac{\gamma-1}{2}q^2 \tag{31}$$

where M_∞ is the free-stream Mach number and the free-stream velocity has been taken to be of unit magnitude. The underlined term in Equation (30) determines when the equation changes type.

A transformation from the (x',y',z')-system to the system (x,y,z) is now made by rotating the coordinate axes. Here x is the free-stream direction and y and z normal to the free stream:

$$\underline{\frac{1}{q^2}(a^2-q^2)[u^2\Phi_{xx} + v^2\Phi_{yy} + w^2\Phi_{zz} + 2uv\Phi_{xy} + 2uw\Phi_{xz} + 2vw\Phi_{yz}]}$$

$$+ \frac{a^2}{q^2}[(q^2-u^2)\Phi_{xx} + (q^2-v^2)\Phi_{yy} + (q^2-w^2)\Phi_{zz}$$

$$- 2uv\Phi_{xy} - 2uw\Phi_{xz} - 2vw\Phi_{yz}] = 0 \tag{32}$$

where $u = \Phi_x$, $v = \Phi_y$ and $w = \Phi_z$ are the velocity components and $q^2 = u^2 + v^2 + w^2$. In Equation (32) the underlined terms result from the underlined term in Equation (30). This equation is no more than a rearrangement of the

standard compressible flow equation for Φ, in which each of the second derivatives has been split into two parts.

Transonic Small-Perturbation Approximation

The transonic small-perturbation equation may be derived from the exact Equation (32) in a number of ways. We give here a derivation in which the scaling of the variables is avoided but the physical meaning of the basic assumptions is stressed.

The total velocity potential Φ is first replaced by a perturbation potential ϕ defined by

$$\Phi = x + \phi \tag{33}$$

where $U_\infty = 1$. The velocity components are now

$$u = 1 + \phi_x \quad , \quad v = \phi_y \quad , \quad w = \phi_z \quad . \tag{34}$$

We assume first that ϕ_x, ϕ_y and ϕ_z are all small and if the z-axis is taken in the vertical direction the slope of the wing-body surface relative to the horizontal plane z = constant is approximately given by $\frac{\partial z}{\partial x} \simeq \frac{w}{u}$. We shall assume that $\frac{w}{u}$ is small:

$$\frac{w}{u} \sim \delta \quad , \quad \delta << 1 \quad . \tag{35}$$

This small-perturbation approximation cannot obviously be valid near a blunt leading edge but, as in two-dimensional sections, we accept this local inconsistency and proceed. It is convenient to choose the streamwise chord length to be of unit length. So the condition $\phi_x << 1$ implies that $\phi << 1$ and we write

$$\phi \sim \varepsilon \quad , \quad \varepsilon << 1 \quad . \tag{36}$$

Now we introduce two lengths b and t in the y and z directions, respectively, such that

$$\phi_y \sim \frac{\varepsilon}{b} \quad , \quad \phi_z \sim \frac{\varepsilon}{t} \quad . \tag{37}$$

The length b is related to the planform of the wing. Thus for any wing of aspect ratio ~ 1 or of any aspect ratio but appreciable sweep, $b \sim 1$. On the other hand, for a wing of high aspect ratio and small sweep, $b >> 1$. This is consistent with the exact relation $\phi_y = - \phi_x \tan\Lambda$ for an infinite swept wing at an angle Λ with constant chord length. The length t is related to the wing slope $\frac{w}{u} \sim \delta$, since $\phi_x = \frac{w}{u}$ approximately, so that

$$\frac{\varepsilon}{t} \sim \delta \quad . \tag{38}$$

The next step is to expand the terms in Equation (32) in the new dependent variable ϕ, by making use of the conditions:

$$\left.\begin{aligned} &\phi \sim \varepsilon << 1 \\ &\phi_y \sim \frac{\varepsilon}{b} << 1 \\ &\phi_z \sim \frac{\varepsilon}{t} << 1 \end{aligned}\right\} \tag{39}$$

Thus, on omitting quantities of third and higher order, we have Equations (31), (32), (34) and (39):

$$\underline{[1-M_\infty^2-(\gamma+1)M_\infty^2\phi_x]\phi_{xx} + 2(1-M_\infty^2)[\phi_y\phi_{xy}+\phi_z\phi_{xz}]}$$
$$+ [1-(\gamma-1)M_\infty^2\phi_x](\phi_{yy}+\phi_{zz}) - 2\phi_y\phi_{xy} - 2\phi_z\phi_{xz} = 0 \tag{40}$$

To the first-order, the pressure coefficient is given by

$$C_p = - 2\phi_x \tag{41}$$

The small-perturbation Equation (40) is now further simplified by making a transonic flow approximation. We suppose that $1 - M_\infty^2 \sim \phi_x$, that is

$$1 - M_\infty^2 \sim \varepsilon \quad . \tag{42}$$

Thus the second of the underlined terms in Equation (40) can be omitted. Furthermore the term $(\gamma-1)M_\infty^2\phi_x(\phi_{yy}+\phi_{zz}) \sim \varepsilon^2$ can also be omitted. The result is the transonic small-disturbance equation for arbitrary wing-body combination:

$$\underline{[1-M^2-(\gamma+1)M^2\phi_x]\phi_{xx}} + \phi_{yy} + \phi_{zz} - 2\phi_y\phi_{xy} - 2\phi_z\phi_{xz} = 0 \tag{43}$$

This equation differs from the commonly adopted equation in having additional terms $2\phi_y\phi_{xy}$ and $2\phi_z\phi_{xz}$.

Note that for $b \sim 1$, that is, for wings of aspect ratio 1, or for wings of any aspect-ratio but appreciable sweep, $(\phi_{yy} + \phi_{zz}) << \varepsilon$. Yet $\phi_{yy} \sim \varepsilon$, so it follows that $\phi_{zz} \sim \varepsilon$ and the length $t \sim 1$. However, for $b \to \infty$, that is, for two-dimensional flows, $\phi_{yy} \to 0$ and $\phi_y\phi_{xy} \to 0$, while $\phi_z\phi_{xz}(\sim\frac{\varepsilon^2}{t^2}) << \phi_{zz}(\sim\frac{\varepsilon}{t^2})$, and the equation reduces to the well-known two-dimensional nonlinear transonic form:

$$[1-M^2-(\gamma+1)M_\infty^2\phi_x]\phi_{xx} + \phi_{xx} = 0 \tag{44}$$

which implies that $\phi_{zz} \sim \varepsilon^2$ so that $t \sim \varepsilon^{-1/2}$, that is $t >> 1$. From the relations (36) and (38) it follows that $\phi \sim t\delta$ and is larger for a two-dimensional airfoil than for a finite wing of the same thickness.

The magnitudes of various quantities for three- and two-dimensional wings follow.

MAGNITUDES OF VARIOUS QUANTITIES FOR
THREE- AND TWO-DIMENSIONAL WINGS

Quantity	3-Dim. Wing	2-Dim. Wing
b: transverse y-direction length scale	1	∞
t: length scale in z-direction	1	$\delta^{-1/3}$
ϕ: perturbation potential	δ	$\delta^{2/3}$
$1-M_\infty^2$	δ	$\delta^{2/3}$
ϕ_x, ~ pressure coefficient	δ	$\delta^{2/3}$
ϕ_{xx}, streamwise pressure gradient	δ	$\delta^{2/3}$
ϕ_{zz}, vertical pressure gradient	δ	δ
$\phi_x\phi_{xx}$ (terms in governing differential equations)	δ^2	$\delta^{4/3}$
$(\phi_{yy}+\phi_{xz})$ (terms in governing differential equations)	δ^2	$\delta^{4/3}$
$\phi_z\phi_{xz}$ (terms in governing differential equations)	δ^2	δ^2

Comments

For the two-dimensional airfoil, the magnitudes that have emerged for t, ϕ and $1-M_\infty^2$ show that if new scaled quantities of unit order of magnitude are to be defined then the appropriate scaling is obtained by writing

$$\left.\begin{aligned} \bar{z} &\sim \delta^{1/3} \\ \bar{\phi} &\sim \delta^{-2/3} \\ K &\sim \frac{1-M_\infty^2}{\delta^{2/3}} \end{aligned}\right\} \qquad (45)$$

Equation (44) then reduces to

$$[K - (\gamma+1)\bar{\phi}_{\bar{x}}]\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} = 0 \tag{46}$$

For a three-dimensional wing the corresponding scaled quantities are different. They are

$$\left.\begin{array}{l} \bar{z} \sim z \\ \bar{\phi} \sim \delta^{-1}\phi \\ K \quad \dfrac{1-M_\infty^2}{\delta} \end{array}\right\} \tag{47}$$

It is very important to recognize these differences. Note that from inspection of Equation (43) it does not seem possible to define similarity parameters like K such that the solution depends only on K (as with two-dimensional airfoils) and not on the thickness δ as well. In other words, there does not appear to be a simple transonic similarity rule for finite wings as there is for airfoils.

The differences in the vertical length scale t and the perturbation potential ϕ imply differences in the physical flow fields. The pressures and streamwise pressure gradients on the wing surface are smaller by a factor of $\sim \delta^{1/3}$ than those on the corresponding airfoil surface. For a finite wing the normal and streamwise pressure gradients are of the same order of magnitude, but for an airfoil section the normal gradients are smaller than the streamwise gradients by a factor of $\sim \delta^{1/3}$.

SECTION III

SUITABILITY OF THE VARIOUS TRANSONIC EQUATIONS

We have already derived the NASA Ames equation given by Lomax, Bailey and Ballhaus (Reference 2):

$$[(1-M_\infty^2)-(\gamma+1)M_\infty^2\phi_x]\phi_{xx}+[1-(\gamma-1)M_\infty^2\phi_x]\phi_{yy}+\phi_{zz}-2M_\infty^2\phi_y\phi_{xy}=0 \tag{48}$$

This equation was suggested for use in the case of flow over finite wing-body combinations where oblique shocks at moderate to large sweep angles are anticipated. If δ is the thickness-to-chord ratio, the assumptions leading to this equation are:

$$\phi = O(\delta^{2/3}) \quad ,$$

$$\text{z-scaling according to } z = \delta^{-1/3}\bar{z} \quad , \tag{49}$$

and $(1 - M_\infty^2) = O(1)$.

For flow with shocks we have

$$M^2\cos^2\theta-1 = M_\infty^2\cos^2\theta-1 + 2M_\infty^2(1+\frac{\gamma-1}{2}M_\infty^2)\phi_x\cos^2\theta + \ldots > 0 \tag{50}$$

where M is the local Mach number in front of the shock and θ the shock sweep angle. Now we have $\phi_x = O(\delta^{2/3}) > 0$ and consequently for $M_\infty < 1$ Equation (50) requires

$$M_\infty^2\cos^2\theta - 1 = O(\delta^p) \quad , \quad p \geq 2/3$$

or

$$M_\infty^2 - 1 = \tan^2\theta + O(\delta^p)\cos^2\theta \quad , \quad p \geq 2/3 \tag{51}$$

whence

$$M_n^2 - 1 \cong M^2 \cos^2\theta - 1 = O(\delta^{2/3}) \ .$$

The ignored entropy rise across a shock is then of order $(M_n-1)^3 \sim (\delta^{2/3})^3 = \delta^2$, which is consistent with the fact that Equation (48) is a second order accurate approximation.

Now since we expect normal shocks to occur in any case, we may have $\theta = 0$ and thus

$$M_\infty^2 - 1 = O(\delta^p) \quad , \quad p \geq 2/3 \quad ,$$

which contradicts the assumption $1 - M_\infty^2 = O(1)$. This means that Equation (48) can only be used for subcritical flow; that is to low transonic speed regime. Yet, it is a nonlinear equation and as such capable of producing shocks.

The classical Guderley-Karman equation is

$$[1-M_\infty^2-(\gamma+1)M_\infty^2\phi_x]\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \tag{52}$$

which is usually given for a slender body as well as for high aspect-ratio wings of very small sweep and dihedral. The necessary assumptions are

$$\phi = O(\delta^{2/3}) \quad ;$$

y- and z-scaling is according to

$$y = \delta^{-1/3}\bar{y} \quad ; \quad z = \delta^{-1/3}\bar{z} \tag{53}$$

and

$$1 - M^2 = O(\delta^{2/3})$$

whence we attribute it to the medium transonic speed regime since M_∞ is closer to one than in the foregoing case. In the derivation of this equation no contradictions are encountered for the case of flow with shocks as long as they are slightly swept.

Newman and Klunker (Reference 6) have added one third order term to the Guderley-Karman equation for a better approximation of the critical speed where the equation changes type from elliptic to hyperbolic:

$$[(1-M_\infty^2)-(\gamma+1)M_\infty^2\phi_x - \frac{1}{2}(\gamma+1)M_\infty^2\phi_x^2]\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad (54)$$

The necessary assumptions are the same as given in Equation (53).

In Reference 7 (see also Reference 8), Hall and Firmin have derived and used the equation (see Section 4):

$$[(1-M_\infty^2)-(\gamma+1)M_\infty^2\phi_x]\phi_{xx} + \phi_{yy} + \phi_{zz} - 2\phi_y\phi_{xy} - 2\phi_z\phi_{xz} = 0 \qquad (55)$$

for a low aspect-ratio wing fuselage combination or a wing of any aspect-ratio but appreciable sweep. The necessary assumptions are

$$\phi = O(\delta) \quad , \quad 1 - M_\infty^2 = O(\delta) \qquad (56)$$

whence we attribute it to the high transonic speed regime since M_∞ is still closer to one than in the foregoing case. Also in the derivation of this equation no contradictions are encountered for the case of flow with shocks.

The Conservation Forms of Small Perturbation Transonic Equations

The mass conservation law for the compressible fluid flow may be written exactly as:

$$[\rho(1+\phi_x)]_x + [\rho\phi_y]_y + [\rho\phi_z]_z = 0 \qquad (57)$$

Various approximations to Equation (57) may be derived by substituting the properly truncated series for the terms inside the brackets:

$$\rho(1+\phi_x) = 1 + (1-M_\infty^2)\phi_x - \frac{1}{2}\left\{3-(2-\gamma)M_\infty^2\right\}M_\infty^2\phi_x^2$$

$$- \frac{1}{2} M_\infty^2\phi_y^2 - \frac{1}{2} M_\infty^2\phi_z^2 + \ldots \qquad (58)$$

$$\rho\phi_y = \phi_y - M_\infty^2\phi_x\phi_y + \ldots$$

$$\rho\phi_z = \phi_z - M_\infty^2\phi_x\phi_z + \ldots$$

in Equation (57). For instance, keeping only terms up to the second powers of ϕ-derivatives in the x- and y-directions the conservation form of the transonic equation given by van der Vooren and his associates at NLR in Holland

$$\left[(1-M_\infty^2)\phi_x - \frac{1}{2}\left\{3-(2-\gamma)M_\infty^2\right\}M_\infty^2\phi_x^2 - \frac{1}{2}M_\infty^2\phi_y^2\right]_x + \left[\phi_y - M_\infty^2\phi_x\phi_y\right]_y + \left[\phi_z\right]_z = 0$$

$$(59)$$

The conservation form of the NASA Ames Equation (48) turns out to be

$$\left[(1-M_\infty^2)\phi_x - \frac{1}{2}(\gamma+1)M_\infty^2\phi_x^2 + \frac{1}{2}(\gamma-3)M_\infty^2\phi_y^2\right]_x$$

$$+ \left[\phi_y - (\gamma-1)M_\infty^2\phi_x\phi_y\right]_y + \left[\phi_z\right]_z = 0 \qquad (60)$$

The Equation (59) can be shown to differ from Equation (60) only by third order terms. However their shock relations differ significantly and will be compared below.

Investigations of Shock Relations

Consider three typical cases of vertical shocks, for which $\phi_z = 0$.

The first case is the normal shock which occurs, for example, at the wing roots. We have $\phi_y = \phi_z = 0$. The

results are shown in Figure 7 for $0.8 \leq M_\infty \leq 1$. The NASA Ames, Guderley-Karman and RAE equations [Figure 7(a)] show fair agreement with full potential equation.

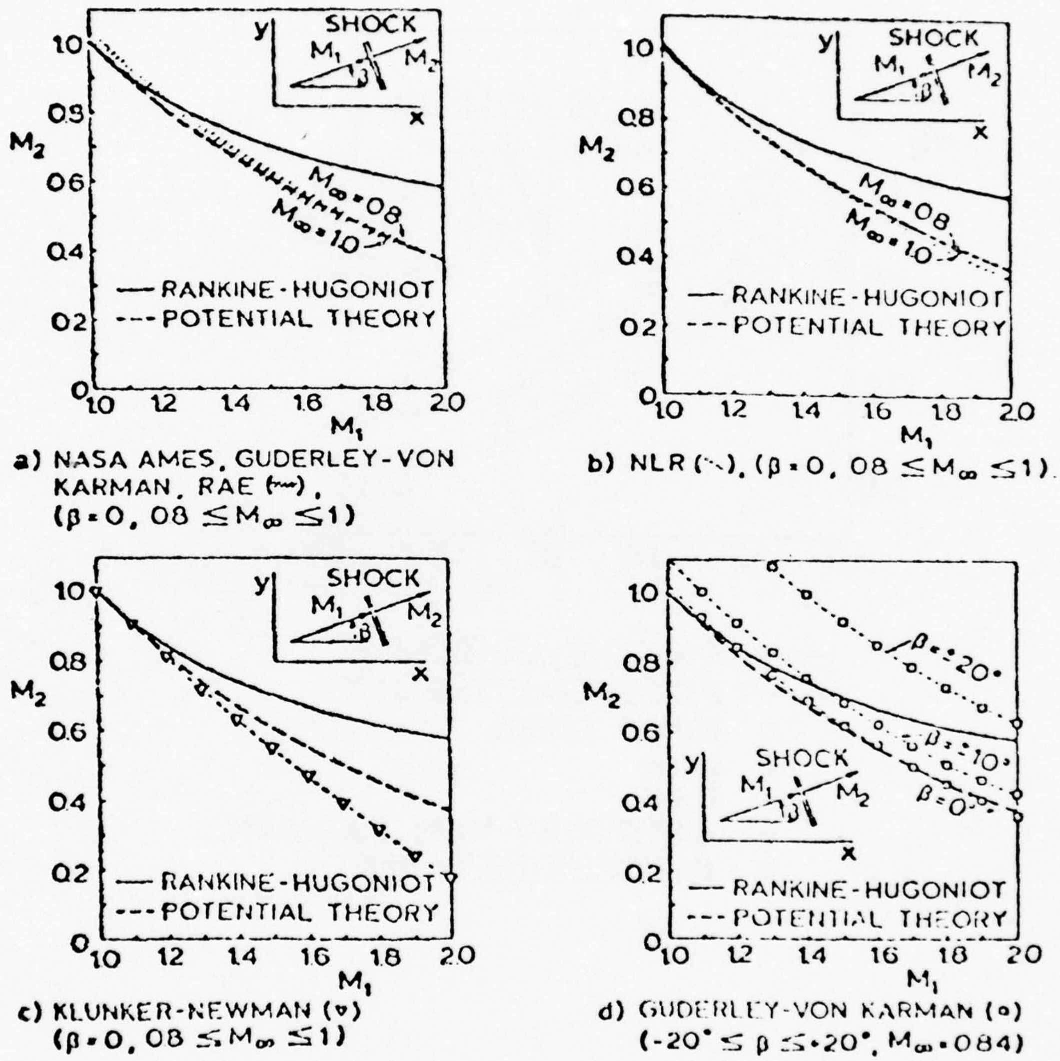


Figure 7. Normal Shock Relations for Various Transonic Equations.

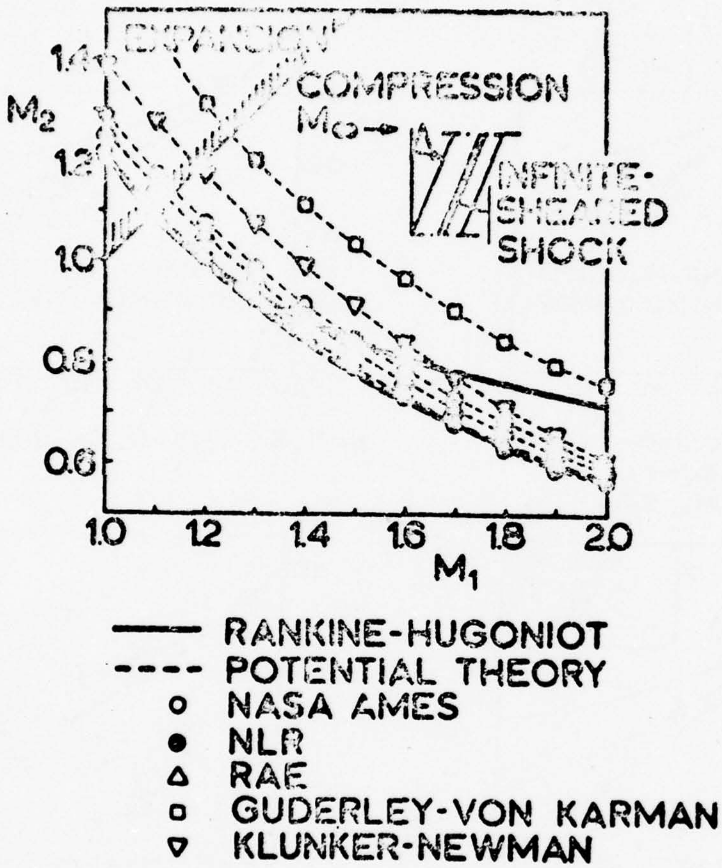


Figure 8(a). Infinite-Sheared Shock Relations for Various Transonic Small Perturbation Equations ($M_\infty = 0.84$, $\Lambda = 30^o$).

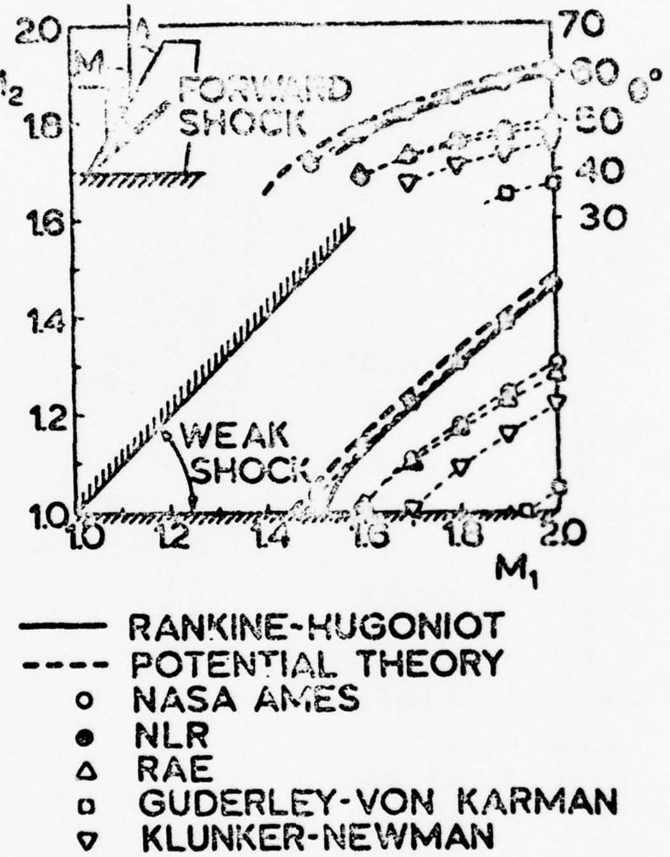


Figure 8(b). Shock Relations for Various Transonic Small Perturbation Equations in a Case of Forward Shock ($M_\infty = 0.84$, $\Lambda = 30^{\circ}$).

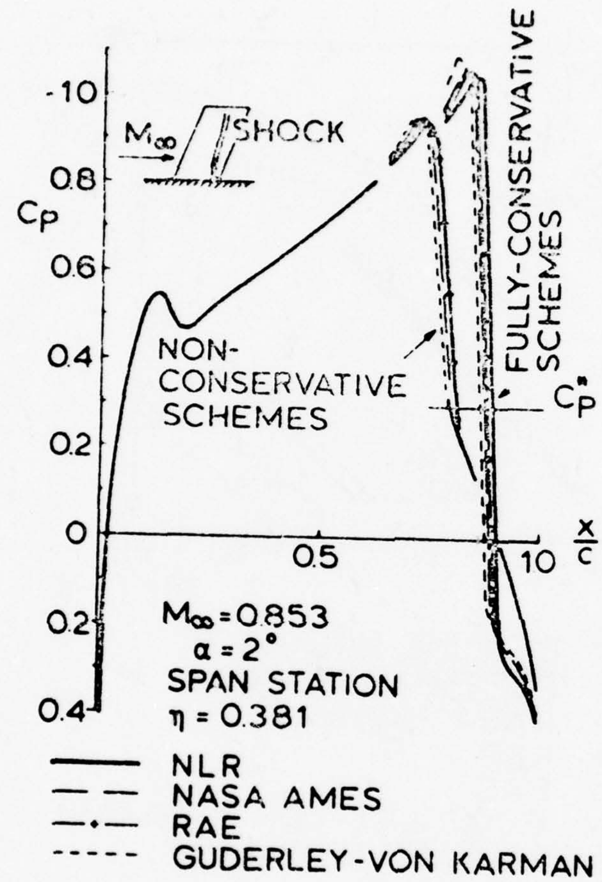
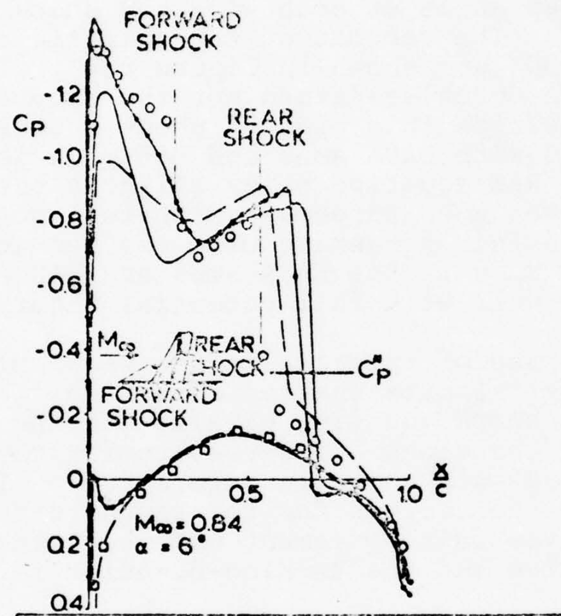


Figure 8(c). Upper Surface Pressure on C141 Panelmodel for Various Transonic Small Perturbation Equations.



MEASURED DATA	SURFACE	SPAN STATION
o	UPPER	$\eta = 0.44$
o	LOWER	

NLR COMP/GRID NEAR NOSE	DIFFERENCE SCHEMES	SPAN STATION
— — — / TOO COARSE	NON CONSERVATIVE	$\eta = 0.381$
——— / TOO COARSE	FULLY CONSERVATIVE	$\eta = 0.381$
—•—•— / VERY FINE	FULLY CONSERVATIVE	$\eta = 0.476$

Figure 8(d). Pressure Distribution on ONERA M6 Wing Model.

For values of $M_1 \leq 1.3$ there is also fair agreement with the Rankine-Hugoniot relations. The results of NLR equation [Figure 7(b)] are practically independent of M_∞ and agree well with the potential theory. The results of the Klunker-Newman equation [Figure 7(c)] are nearly independent of M_∞. Though the agreement with full potential theory as well as with the Rankine-Hugoniot relations is good for low M_1-values the overall picture is not as good as for the NLR equation, or even the other three equations for reasons already indicated.

Secondly, the strong oblique shock on an infinite sheared wing, for the various equations, was compared. If Λ is the sweep angle of both wing and shock we have $\phi_y = - \phi_x \tan \Lambda$. The results of the specific case $M_\infty = 0.84$, $\Lambda = 30^o$ are shown in Figure 8(a). It is apparent that neither the Guderley-Karman nor the Klunker-Newman equation can describe this kind of shock adequately. The results obtained with NASA Ames and RAE equations are reasonable, the RAE equation being slightly better. The NLR equation gives good agreement with full potential theory and for $M_1 \leq 1.3$ fair agreement with the Rankine-Hugoniot relations. For $M_\infty = 1$, the NASA Ames and RAE equations also agree very well with full potential theory.

The third case of interest is the weak oblique forward shock. Assuming infinite-sheared conditions ($\phi_y = - \phi_x \tan \Lambda$) in front of the shock and flow parallel to the fuselage ($\phi_y = 0$) behind the shock. The results for the case $M_\infty = 0.84$, $\Lambda = 30^o$ are shown in Figure 8(b). The conclusions are the same as for the second case. Only the NLR equation gives fair agreement with both the full potential equation and the Rankine-Hugoniot relations.

In Reference 9, van der Vooren and his associates have given some results of calculations of transonic flows over isolated semi-wings. Nonconservative as well as fully-conservative rotated difference schemes have been used. They came to the conclusion that a nonconservative scheme is inadequate for shock capturing capability [see Figures 8(c) and 8(d)].

SECTION IV

INTEGRAL EQUATION FOR THREE-DIMENSIONAL TRANSONIC FLOWS PAST WING-BODY COMBINATION

THE NONLIFTING CASE

Theoretical pressure distributions on nonlifting circular arc airfoils immersed in a high subsonic free stream were first given by Oswatitsch in 1950 by approximately solving the integral equation, which was proposed by him. The analysis is carried out in the physical rather than in the hodograph variables, and leads to a nonlinear integral equation in which the unknown velocity appears outside as well as inside the integral. Oswatitsch found approximate solutions not by iteration, but by introducing various functions containing undetermined parameters into the integral equation and then by determining these unknowns by satisfying the integral equation at a small number of points on the airfoil chord. The application of the method to some airfoil section did show certain definite characteristics of transonic flow such as the appearance of shock waves and their rearward movement across the chord with increasing Mach number. However, the method fails to give proper results at high subsonic M_∞ greater than about 0.88 with a 6 percent-thick circular arc airfoil.

Gullstrand, at KTH, Sweden, tried to rectify the situation by seeking a solution by iteration. Gullstrand uses the integral equation to determine only the solution for the forward part of the airfoil and then uses the method of characteristics to complete the solution for the rear of the airfoil.

Spreiter and Alksne (Reference 10) proposed in 1955 a solution of the three-dimensional integral equation given by Oswatitsch using an iteration process permitting the integral equation to be satisfied at a much larger number of points than in the original method of Oswatitsch. This method gives approximate solutions at all Mach numbers up to unity, and appears to avoid any multiplicity of solutions that were unavoidable in Oswatitsch's earlier method for supercritical Mach numbers.

Basic Equations

The simplified small perturbation equation in the transonic flow regime is the usual nonlinear equation

$$(1-M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = K\ \phi_x\phi_{xx} \tag{61}$$

where

$$k = M_\infty^2\ \frac{\gamma+1}{U_\infty} \tag{62}$$

and ϕ , the perturbation velocity potential given by

$$\phi = -\ U_\infty x + \Phi$$

As a result of minor variations in the perturbation analysis, other authors have used at least four different relations for k, the coefficient of the nonlinear term in the simplified equation for the transonic flow. A straightforward development of the second-order theory leads to the relation given in Equation (62). This is sometimes simplified to

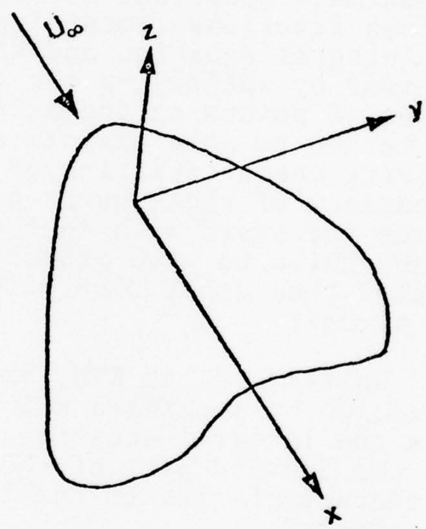


Figure 9. View of Wing and the Coordinate System.

$$k = \frac{\gamma+1}{U_\infty} \tag{63}$$

by arguing that M_∞ can be set equal to unity since the right-hand side is after all an approximation to the actual second-order equation. If the full nonlinear equation in Φ ,

$$(a^2-u^2)\phi_{xx} - 2uv\phi_{xy} + (a^2-v^2)\phi_{yy} = 0$$

the total velocity potential is divided by a^2 and the quotient $1/a^2$ in each term is expanded in a binomial series, the resulting coefficient k of the term $\phi_x\phi_{xx}$ turns out to be

$$k = M_\infty^2 \left[\frac{2 + (\gamma-1)M_\infty^2}{U_\infty}\right] \tag{64}$$

Still another expression for k is used by Oswatitsch by writing

$$1 - M^2 = 1 - M_\infty^2 - \frac{1-M_\infty^2}{a^*-U_\infty}\phi_x + \ldots$$

thereby giving the value of k as

$$k = \frac{1-M_\infty^2}{a^*-U} \tag{65}$$

where a* is the critical sound speed given by

$$\frac{a^*}{U_\infty} = \sqrt{\frac{\gamma-1}{\gamma+1} + \frac{2}{M_\infty^2(\gamma+1)}}$$

A similar situation arises in the derivation of the simplified equation for the shock polar given below. A significant case where the four relations for k lead to different results is the prediction of the variation of the critical pressure coefficient $C_{p_{cr}}$ with M_∞. The critical pressure coefficient $C_{p_{cr}}$ is the value of C_p at a point where local M = 1. This condition is recognized by the vanishing of the coefficient of ϕ_{xx}, thus

$$1 - M_\infty^2 - k(\phi_x)_{cr} = 0$$

or

$$C_{p_{cr}} = -\frac{2}{U_\infty}(\phi_x)_{cr} = -\frac{2}{kU_\infty}(1-M_\infty^2) \tag{66}$$

The exact relation for isentropic flow from Reference 1 is

$$C_{p_{cr}} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} M^2\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

Similar comparisons can be made for local Mach numbers M other than unity or by considering the velocity jump across a shock wave or by comparison to some of the existing experimental results on drag of a wedge. Each of these comparisons provides striking evidence in support of the value of k given by Equation (62) although in some cases Oswatitsch's value, Equation (65), gave good results as well.

Equation (61) is valid only in regions where the necessary derivations exist and are continuous. Since these conditions do not hold where shock waves occur, an additional equation is needed for the transition through the shock. The necessary equation is provided by the classical regulation for the shock polar:

$$\tilde{v}_2^2 + \tilde{w}_2^2 = (\tilde{u}_1-\tilde{u}_2)^2 \frac{\tilde{u}_1\tilde{u}_2 - a^{*2}}{\frac{2}{\gamma+1}\tilde{u}_1^2 - \tilde{u}_1\tilde{u}_2 + a^{*2}} \tag{67}$$

where $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ refer to Cartesian velocity components with $\tilde{u}$ being parallel to the flow direction ahead of the shock, the subscripts 1 and 2 refer to conditions ahead of and behind the shock, and a^* is the critical sound speed. Carrying out a small perturbation analysis of Equation (67) analogous to that performed in the derivation of Equation (61), it can be shown that the following relation results between the perturbation velocity components on the two sides of the shock wave:

$$(1-M_\infty^2)(u_1-u_2)^2 + (v_1-v_2)^2 + (w_1-w_2)^2 = k\left(\frac{u_1+u_2}{2}\right)(u_1-u_2)^2 \qquad (68)$$

where now u, v and w are the perturbation velocity components parallel to x, y and z-axes. This equation corresponds to the shock-polar curve for shock saves of small strength inclined at any angle between that of normal shock waves and that of the Mach lines. On either side of the shock wave, Equation (61) holds.

The Boundary Conditions

(a) At $x = -\infty$

$$(\phi_x)_\infty = (\phi_y)_\infty = (\phi_z)_\infty = 0 \qquad (69)$$

(b) At the wing or body surface

$$\frac{1}{U_\infty}(\phi_z)_{W \text{ or } B} = \frac{\partial z}{\partial x} \qquad (70)$$

where $\partial z/\partial x$ is the local slope of the wing or body surface in the x-direction.

In addition, it is necessary to prescribe that the direct influence of a disturbance in the supersonic region proceeds only in the downstream direction and that the Kutta condition applies whenever the flow velocity at the trailing edge is subsonic for unique solution.

Integral Equation for Transonic Flow (Non-Lifting Case)

Since the principal object of the following analysis is to determine the perturbation velocity components at any point, it is convenient to work with equivalent equations for u, v and w obtained by differentiating Equation (61):

$$(1-M_\infty^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = k\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}$$

with respect to x, y and z, respectively:

$$(1-M_\infty^2)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = k\frac{\partial^2}{\partial x^2}\left(\frac{u^2}{2}\right) \tag{71}$$

$$(1-M_\infty^2)\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = k\frac{\partial^2}{\partial y\partial x}\left(\frac{u^2}{2}\right) \tag{72}$$

$$(1-M_\infty^2)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = k\frac{\partial^2}{\partial z\partial x}\left(\frac{u^2}{2}\right) \tag{73}$$

since

$$\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} = u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) .$$

We will assume that $M_\infty < 1$, i.e., the free stream is a high subsonic velocity though the local speeds on the body exceed unity in some areas.

Then normalizing Equations (71), (72) and (73) by letting

$$\bar{x} = x \quad , \quad \bar{y} = \beta y \quad , \quad \bar{z} = \beta z \quad , \quad \bar{\phi} = \frac{k}{\beta^2}\phi$$

$$\bar{u} = \frac{\partial\bar{\phi}}{\partial\bar{x}} = \frac{k}{\beta^2}u \quad , \quad \bar{v} = \frac{\partial\bar{\phi}}{\partial\bar{y}} = \frac{k}{\beta^3}v \quad , \quad \bar{w} = \frac{\partial\bar{\phi}}{\partial\bar{z}} = \frac{k}{\beta^3}w \tag{74}$$

where

$$\beta = \sqrt{1 - M_\infty^2}$$

Equations (61) and (71), (72) and (73) reduce to the following:

$$\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{y}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{z}^2} \equiv \nabla^2\bar{\phi} = \frac{\partial\bar{\phi}}{\partial\bar{x}}\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} \tag{75}$$

$$\nabla^2\bar{u} = \frac{\partial^2}{\partial\bar{x}^2}\left(\frac{\bar{u}^2}{2}\right) \tag{76}$$

$$\nabla^2\bar{v} = \frac{\partial^2}{\partial\bar{y}\partial\bar{x}}\left(\frac{\bar{u}^2}{2}\right) \tag{77}$$

$$\nabla^2\bar{w} = \frac{\partial^2}{\partial\bar{z}\partial\bar{x}}\left(\frac{\bar{u}^2}{2}\right) \tag{78}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^2}{\partial\bar{y}^2} + \frac{\partial^2}{\partial\bar{z}^2}$$

is the Laplacian operation in the normalized variables $\bar{x}$, $\bar{y}$ and $\bar{z}$.

Before proceeding, it should be noted that the introduction of the reduced perturbation velocity component $\bar{u}$ permits the ready recognition of regions of subsonic and supersonic velocities and emphasizes the points at which sonic velocity occurs. This becomes apparent from the approximate relation between the local M and ϕ_x:

$$1 - M^2 = 1 - M_\infty^2 - k\,\phi_x \tag{79}$$

or

$$\frac{1 - M^2}{1 - M_\infty^2} = 1 - \frac{k}{1 - M_\infty^2}u = 1 - \bar{u}$$

from which it is clear, for flows having subsonic free-stream $M_\infty(M_\infty < 1)$, that $\bar{u} < 1$ when the local velocity is subsonic, $\bar{u} = 1$ when it is sonic, and $\bar{u} > 1$ when it is supersonic.

We will now apply Green's theorem to Equation (76). The results for Equations (77) and (78) will follow immediately. If σ and Ω are any two functions which, together with their first and second derivatives, are finite and single valued throughout a region R enclosed by a surface Σ, Green's theorem, which follows from Gauss' divergence theorem, states:

$$\iint_\Sigma \left(\sigma \frac{\partial\Omega}{\partial n} - \Omega \frac{\partial\sigma}{\partial n}\right) d\Sigma = \iiint_R \left(\Omega\nabla^2\sigma - \sigma\nabla^2\Omega\right) dR \tag{80}$$

where the directional derivatives on the left-hand side are taken along the normal n, drawn inward, to the surface.

Now in Equation (80) let $\Omega = \bar{u}$ and choose σ as the fundamental solution $1/r_3$ of the Laplace equation $\nabla^2\sigma = 0$, i.e.,

$$\sigma = \frac{1}{r^3} = \frac{1}{\left[(\bar{x}-\xi)^2+(\bar{y}-\eta)^2+(\bar{z}-\zeta)^2\right]^{1/2}} . \tag{81}$$

Then we have from Equation (80)

$$\iint_\Sigma \left[\frac{1}{r^3} \frac{\partial\bar{u}}{\partial n} - \bar{u} \frac{\partial}{\partial n} \left(\frac{1}{r_3}\right)\right] d\Sigma = - \iiint_R \frac{1}{r_3} \nabla^2 \bar{u} \, dR$$

$$= - \iiint_R \frac{1}{r^3} \frac{\partial^2}{\partial\xi^2} \left(\frac{\bar{u}^2}{2}\right) dR \tag{82}$$

since from Equation (76)

$$\nabla^2\bar{u} = \frac{\partial^2}{\partial\bar{x}^2} \left(\frac{\bar{u}^2}{2}\right)$$

The variables of integration in Equation (82) are $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ while $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of a point P. It must be observed that $1/r_3$ is singular at $r_3 = 0$ and $\bar{u}$ is discontinuous at the shock wave. The point P and the shock must, therefore, be excluded from the region R. A schematic indication of the body (wing or fuselage) and the region of integration is shown in Figure 10. The complete three-dimensional extent of the body has not been pictured. It suffices, however, to state that the surface Σ (shown dashed) is composed of a sphere of large radius which forms the external boundary of R, an infinitesimal sphere surrounding P, and a final surface enveloping the wing and body, the wake (for the lifting case), and the shock waves.

Let us now apply Equation (82) to the region R_u bounded by the $\overline{xy}$-plane and a hemispherical dome of infinite radius lying above this plane, exclusive of the subregions surrounding $P(\bar{x},\bar{y},\bar{z})$ and the shock waves (Figure 10). Since, furthermore, $\bar{u}$ may be assumed to diminish sufficiently rapidly with distance, the contributions of the integrals over the hemisphere vanish and the contribution of the surface integral over the small sphere surrounding P may easily be shown to be $4\pi\ \bar{u}(\bar{x},\bar{y},\bar{z})$ when the radius $\varepsilon_1 \to 0$. We have, therefore, from Equation (82) the following result:

$$\bar{u}(\bar{x},\bar{y},\bar{z}) = -\frac{1}{4\pi}\iint_{+\infty}^{-\infty}\left[\frac{1}{r_3}\frac{\partial \bar{u}_u}{\partial \bar{\zeta}} - \bar{u}_u\frac{\partial}{\partial \bar{\zeta}}\left(\frac{1}{r_3}\right)\right]d\bar{\xi}d\bar{\eta}$$

$$-\frac{1}{4\pi}\iint_{S_u}\left[\left\{\frac{1}{r_3}\frac{\partial \bar{u}}{\partial n} - \bar{u}\frac{\partial}{\partial n}\left(\frac{1}{r_3}\right)\right\}_1 + \left\{\frac{1}{r_3}\frac{\partial \bar{u}}{\partial n} - \bar{u}\frac{\partial}{\partial n}\left(\frac{1}{r_3}\right)\right\}_2\right]dS$$

$$-\frac{1}{4\pi}\iiint_{R_u}\frac{1}{r_3}\frac{\partial^2}{\partial \bar{\xi}^2}\left(\frac{\bar{u}^2}{2}\right)dR \qquad (83)$$

where the subscript u denotes conditions on the upper side of the $\overline{xy}$-plane, the subscripts 1 and 2 denote values immediately ahead of and behind the shock wave, and S is the surface of the shock wave. The volume integral is defined as follows when P is ahead of S:

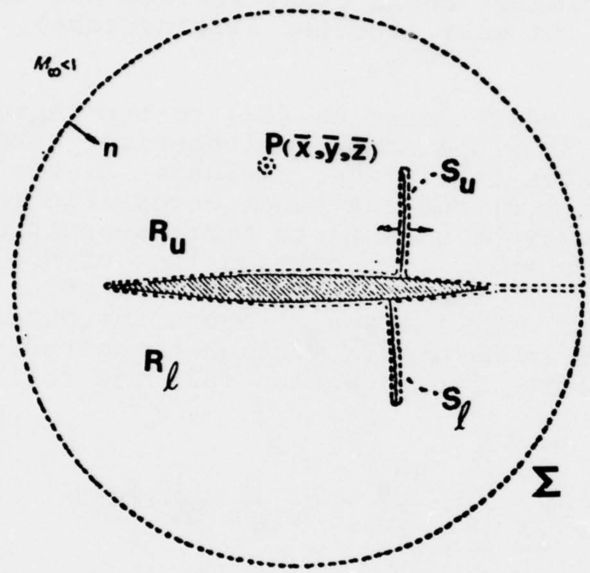


Figure 10. Region of Integration: $M_\infty \leq 1$.

$$\iiint_{R_u} \frac{1}{r_3} \frac{\partial^2}{\partial\bar{\xi}^2} \left(\frac{\bar{u}^2}{2}\right) dR \equiv \iiint_{R_u} \psi dR =$$

$$\lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \int_{-\infty}^{\infty} d\bar{\eta} \int_0^{+\infty} d\bar{\zeta} \left\{ \int_{-\infty}^{\bar{x}_p - \varepsilon_1} \psi d\bar{\xi} + \int_{\bar{x}_p + \varepsilon_1}^{\bar{x}_S - \varepsilon_2} \psi d\bar{\xi} + \int_{\bar{x}_S + \varepsilon_2}^{+\infty} \psi d\bar{\xi} \right\} \tag{84}$$

where we have written $\psi \equiv \frac{1}{r_3} \frac{\partial^2}{\partial\bar{\xi}^2} \left(\frac{\bar{u}^2}{2}\right)$ and ε_1, ε_2 are the the radius of the small sphere surrounding $P(\bar{x},\bar{y},\bar{z})$ and the thickness of the surface surrounding the shock wave surface S_u, respectively. Similarly, when P is behind S, the volume integral is

$$\iiint_{R_u} \psi dR = \lim_{\substack{\varepsilon_1 \to 0 \\ \varepsilon_2 \to 0}} \int_{-\infty}^{+\infty} d\bar{\eta} \int_0^{\infty} d\bar{\zeta} \left\{ \int_{-\infty}^{\bar{x}_s - \varepsilon_2} \psi d\bar{\xi} + \int_{\bar{x}_s - \xi_2}^{\bar{x}_p - \varepsilon_1} \psi d\bar{\xi} + \int_{\bar{x}_p + \varepsilon_1}^{\infty} \psi d\bar{\xi} \right\} \tag{85}$$

If P is kept fixed in the upper half space and the region R bounded by the $\overline{xy}$-plane and a hemispherical dome of infinite radius lying below this plane is considered, it follows in a similar manner that

$$0 = \frac{1}{4\pi} \iint_{-\infty}^{+\infty} \left[\frac{1}{r_3} \frac{\partial \bar{u}_\ell}{\partial \bar{\zeta}} - \bar{u}_\ell \frac{\partial}{\partial \bar{\zeta}} \left(\frac{1}{r_3}\right) \right] d\bar{\xi} d\bar{\eta} - \frac{1}{4\pi} \iint_{S_\ell} \left[\left\{ \frac{1}{r_3} \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial}{\partial n} \left(\frac{1}{r_3}\right) \right\}_1 \right.$$

$$\left. + \left\{ \frac{1}{r_3} \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial}{\partial n} \left(\frac{1}{r_3}\right) \right\}_2 \right] dS - \frac{1}{4\pi} \iiint_{R_\ell} \frac{1}{r_3} \frac{\partial^2}{\partial \bar{\xi}^2} \left(\frac{\bar{u}^2}{2}\right) dR \tag{86}$$

where the subscript ℓ denotes conditions on the lower side of the $\overline{xy}$-plane and the volume integral is defined as

$$\iiint\limits_{R_\ell} \frac{1}{r_3}\frac{\partial^2}{\partial\bar{\xi}^2}\left(\frac{\bar{u}^2}{2}\right)dR \equiv \iiint\limits_{R_\ell} \psi dR = \lim_{\varepsilon_2\to 0}\int_{-\infty}^{+\infty} d\bar{\eta}\int d\bar{\zeta}\left\{\int_{-\infty}^{\bar{x}_s-\varepsilon_2}\psi d\bar{\xi} + \int_{\bar{x}_s+\varepsilon_2}^{+\infty}\psi d\bar{\xi}\right\} \tag{87}$$

Introducing the notation

$$\Delta\bar{u} = \bar{u}_u - \bar{u}_\ell \quad , \quad \Delta\frac{\partial\bar{u}}{\partial\bar{\zeta}} = \frac{\partial\bar{u}_u}{\partial\bar{\zeta}} - \frac{\partial\bar{u}_\ell}{\partial\bar{\zeta}} \tag{88}$$

and adding Equations (83) and (86), we have

$$\bar{u}(\bar{x},\bar{y},\bar{z}) = -\frac{1}{4\pi}\iint\limits_B\left[\frac{1}{r_3}\Delta\frac{\partial\bar{u}}{\partial\bar{\zeta}} - \Delta\bar{u}\frac{\partial}{\partial\bar{\zeta}}\left(\frac{1}{r_3}\right)\right]d\bar{\xi}d\bar{\eta}$$

$$-\frac{1}{4\pi}\iint\limits_S\left[\left\{\frac{1}{r_3}\frac{\partial\bar{u}}{\partial n} - \bar{u}\frac{\partial}{\partial n}\left(\frac{1}{r_3}\right)\right\}_1 + \left\{\frac{1}{r_3}\frac{\partial\bar{u}}{\partial n} - \bar{u}\frac{\partial}{\partial n}\left(\frac{1}{r_3}\right)\right\}_2\right]dS$$

$$-\frac{1}{4\pi}\iiint\frac{1}{r_3}\frac{\partial^2}{\partial\bar{\xi}^2}\left(\frac{\bar{u}^2}{2}\right)dR \tag{89}$$

where B represents the wing-body or wing alone and S represents the sum of the surfaces S_u and S_ℓ; and $R = R_u + R_\ell$. The first integral on the right-hand side of Equation (89) will be recognized as the value of $\bar{u}$ given by the linearized theory of subsonic flow past wing or wing-body combination. Denoting this linear value by $\bar{u}_L$, we have

$$\bar{u}_L = -\frac{1}{4\pi}\iint\limits_B\left[\frac{1}{r_3}\Delta\frac{\partial\bar{u}_L}{\partial\bar{\zeta}} - \Delta\bar{u}_L\frac{\partial}{\partial\bar{\zeta}}\left(\frac{1}{r_3}\right)\right]d\bar{\xi}d\bar{\eta} \tag{90}$$

Equation (89) may be regarded as the final integral equation for $\bar{u}$ for the nonlifting case. But for numerical computations, it is advantageous to write it in another form by integrating the volume integral twice by parts with respect

to $\bar{\xi}$, taking proper cognizance of the definitions given in Equations (84), (85) and (87), and to decompose the surface integral over the shock wave into components parallel to the axes of the coordinate system. If then n_1, n_2 and n_3 are the direction cosines of the normal to the shock surface drawn inward as shown in Figure 10, the following equation is obtained from Equation (89):

$$\bar{u} = \bar{u}_L + \frac{\bar{u}^2}{2} - \frac{1}{4\pi} \iiint_R \frac{\bar{u}^2}{2} \frac{\partial^2}{\partial \bar{\xi}^2} \left(\frac{1}{r_3}\right) d\bar{\xi} d\bar{\eta} d\bar{\zeta}$$

$$+ \frac{1}{4\pi} \iint_S \Bigg[\left\{ \frac{1}{r_3} \frac{\partial}{\partial \bar{\xi}} \left(\bar{u} - \frac{\bar{u}^2}{2}\right) - \left(\bar{u} - \frac{\bar{u}^2}{2}\right) \frac{\partial}{\partial \bar{\xi}} \left(\frac{1}{r_3}\right) \right\}_1$$

$$- \left\{ \frac{1}{r_3} \frac{\partial}{\partial \bar{\xi}} \left(\bar{u} - \frac{\bar{u}^2}{2}\right) - \left(\bar{u} - \frac{\bar{u}^2}{2}\right) \frac{\partial}{\partial \bar{\xi}} \left(\frac{1}{r_3}\right) \right\}_2$$

$$+ \left\{ \left(\frac{1}{r_3} \frac{\partial \bar{u}}{\partial \bar{\eta}} - \bar{u} \frac{\partial}{\partial \bar{\eta}} \left(\frac{1}{r_3}\right) \right)_1 - \left(\frac{1}{r_3} \frac{\partial \bar{u}}{\partial \bar{\eta}} - \bar{u} \frac{\partial}{\partial \bar{\eta}} \left(\frac{1}{r_3}\right) \right)_2 \left(\frac{n_2}{n_1}\right)_2 \right.$$

$$\left. + \left\{ \left(\frac{1}{r_3} \frac{\partial \bar{u}}{\partial \bar{\zeta}} - \bar{u} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{1}{r_3}\right) \right)_1 - \left(\frac{1}{r_3} \frac{\partial \bar{u}}{\partial \bar{\zeta}} - \bar{u} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{1}{r_3}\right) \right)_2 \right. \Bigg] d\bar{\eta} d\bar{\zeta} \tag{91}$$

where $\bar{u}_L$ is given by Equation (90).

Although the integral equation obtained above appears to be more complicated than Equation (89), it is superior from the point of view of approximate solutions for the following reasons:

(a) The triple integral of Equation (89) shows a very strong influence of the velocities in the neighborhood of P since they are multiplied by $1/r_3$. This influence is

largely nullified in the triple integral in Equation (91) because part of the region has a negative influence and part has positive influence. The predominant influence in Equation (91) is furnished by the term $\overline{u}^2/2$ standing outside the integral.

(b) The contribution of distant regions is also diminished in importance in the triple integral of Equation (91) since their influence now varies as $1/r_3^3$ rather than $1/r_3$ as in Equation (89).

(c) The biggest advantage in using Equation (91) is that the value of the triple integral in this equation is continuous through a shock wave (due to the presence of the term $\overline{u}^2/2$ outside the second derivative) rather than discontinuous as in Equation (89). A point of great importance in the approximate solution arises from the fact that the integration by parts provides extra terms (those containing $\overline{u}^2/2$) in the integrals along the shock surface S which combine with those already present in such a way that the contribution of these integrals becomes very small when the shocks are nearly normal waves, as is usually the case for high subsonic speeds.

In addition to satisfying the integral equation for $\overline{u}$ given in Equation (91), the velocity components on opposite sides of shock waves must be in accord with the simplified relation for the shock polar given in Equation (68), which can be rewritten in normalized form, using Equation (74) as follows:

$$(\bar{u}_1 - \bar{u}_2)^2 + (\bar{v}_1 - \bar{v}_2)^2 + (\bar{w}_1 - \bar{w}_2)^2 = \left(\frac{\bar{u}_1 + \bar{u}_2}{2}\right)(\bar{u}_1 - \bar{u}_2)^2 \tag{92}$$

or with slight manipulations it can be written in the following two alternative forms:

$$\left(1 - \frac{\bar{u}_1 + \bar{u}_2}{2}\right)(\bar{u}_1 - \bar{u}_2)^2 + (\bar{v}_1 - \bar{v}_2)^2 + (\bar{w}_1 - \bar{w}_2)^2 = 0 \tag{93}$$

and

$$(\bar{u}_1 - \bar{u}_2)\left[\left(u_1 - \frac{{u_1}^2}{2}\right) - \left(u_2 - \frac{{u_2}^2}{2}\right)\right] + (\bar{v}_1 - \bar{v}_2)^2 + (\bar{w}_1 - \bar{w}_2)^2 = 0 \tag{94}$$

If the shock wave is a normal wave and the flow is parallel to the $\bar{x}$-axis (i.e., $\bar{v}_1 = \bar{v}_2 = \bar{w}_1 = \bar{w}_2 = 0$, but $\bar{u}_1 \neq \bar{u}_2$), it can be seen from Equation (93) that $\bar{u}$ jumps from $1 + \Delta$ immediately ahead of the shock to $1 - \Delta$ immediately behind the shock, where $\Delta = \frac{u_1 - u_2}{2}$. On the other hand, Equation (94) shows that the quantity $\bar{u} - \bar{u}^2/2$ is equal on the two sides of the shock. This is consistent with the fact that the latter quantity corresponds, in the transonic approximation, to the mass flow, which is continuous through a normal shock.

The general problem of the three-dimensional transonic flow about nonlifting bodies requires the solution of Equation (91) while taking account of the shock relations given in Equations (92), (93), and (94). This is a formidable task well beyond the reach of present analysis. We therefore introduce some further approximations before solutions can be obtained. They are: (a) all shock waves are assumed to lie in a plane perpendicular to the x-axis and (b) the shock waves are assumed to be normal to the local flow direction. The first assumption corresponds to setting $n_2 = n_3 = 0$ and the second leads to the relations:

$$\left(\bar{u} = \frac{\bar{u}^2}{2}\right)_1 = \left(\bar{u} - \frac{\bar{u}^2}{2}\right)_2 \quad ; \quad \frac{\partial}{\partial\bar{\xi}}\left(\bar{u} - \frac{\bar{u}^2}{2}\right)_1 = \frac{\partial}{\partial\bar{\xi}}\left(\bar{u} - \frac{\bar{u}^2}{2}\right)_2 = 0 \tag{95}$$

Hence Equation (91) simplifies to

$$\bar{u} = \bar{u}_L + \frac{\bar{u}^2}{2} - \frac{1}{4\pi}\iiint_R \frac{\bar{u}^2}{2}\frac{\partial^2}{\partial\bar{\xi}^2}\left(\frac{1}{r_3}\right)d\bar{\xi}\,d\bar{\eta}\,d\bar{\zeta} \tag{96}$$

A similar, but slightly more involved analysis shows that from Equations (77) and (78), the equations to determine v and w are

$$\bar{v} = \bar{v}_L - \frac{1}{4\pi} \iiint_R \frac{\bar{u}^2}{2} \frac{\partial^2}{\partial\bar{\xi}\partial\bar{\eta}} \left(\frac{1}{r_3}\right) d\bar{\xi} d\bar{\eta} d\bar{\zeta} \tag{97}$$

$$\bar{w} = \bar{w}_L - \frac{1}{4\pi} \iiint_R \frac{\bar{u}^2}{2} \frac{\partial^2}{\partial\bar{\xi}\partial\bar{\zeta}} \left(\frac{1}{r_3}\right) d\bar{\xi} d\bar{\eta} d\bar{\zeta} \tag{98}$$

Before proceeding further, it should be observed that the solutions of Equations (96), (97) and (98) must approach those of the linear theory when $M_\infty << 1$, since $\bar{u} << 1$ and therefore the terms involving $\bar{u}^2$ become negligible, thereby leaving only

$$(\bar{u}, \bar{v}, \bar{w})_{M_\infty << 1} \simeq (\bar{u}_L, \bar{v}_L, \bar{w}_L) \quad . \tag{99}$$

Let us now write Equation (96) as follows:

$$\bar{u} = \bar{u}_L + \frac{\bar{u}^2}{2} - \frac{I}{2} \tag{100}$$

where

$$I = 2\left[\frac{1}{4\pi} \iiint_R \frac{\bar{u}^2}{2} \frac{\partial^2}{\partial\bar{\xi}^2} \left(\frac{1}{r_3}\right) d\bar{\xi} d\bar{\eta} d\bar{\zeta}\right] \tag{101}$$

Although I is a function of $\bar{u}$ and is therefore unknown, it is informative to rewrite Equation (100) by solving for $\bar{u}$ in terms of I and $\bar{u}_L$, thus

$$\bar{u} = 1 \pm \sqrt{I - (2\bar{u}_L - 1)} = 1 \pm \sqrt{I - L} \tag{102}$$

where

$$L = 2\bar{u}_L - 1 \quad .$$

Several interesting observations can be made from Equation (102). First of all, for real values of $\bar{u}$

$$I \geq L \quad . \tag{103}$$

Furthermore, the choice of the plus or minus sign determines whether the local velocity is subsonic or supersonic. A change in sign at a point where the radical is zero corresponds to a discontinuous jump in velocity. As pointed out following Equation (94), such discontinuities correspond to normal shock waves and are permissible when they proceed from supersonic to subsonic velocities or from plus to minus sign in Equation (102), when progressing in the flow direction. Discontinuities in the reverse direction are inadmissable since they correspond to expansion shocks, a phenomenon which violates the second law of thermodynamics.

The values of $\bar{u}_L$, and hence L, can be calculated for any given body and are generally characterized by certain regions in which $\bar{u}_L > 0$ and other regions in which $\bar{u}_L < 0$. The absolute values of $\bar{u}$ increase continuously with increasing M_∞ and the maximum positive values may considerably exceed unity as sonic velocity is approached in the free stream. Not very much can be stated at this point about the values of I, except that they depend on the distribution as well as magnitude of $\bar{u}$ and that the above inequality must be satisfied.

The Two-Dimensional Flow Past Circular Arc Airfoil: The Oswatitsch-Spreiter-Alksne Solution of the Integral Equation

The starting point for the solution of the full integral equation derived for the three-dimensional cases, would be to examine the methods needed for two-dimensional flows given by a number of authors: Oswatitsch, Gullstrand, Spreiter and Alksne, Nixon, and Norstrud.

The analysis of the two-dimensional case is based on the following equation:

$$\bar{u}(\bar{x},\bar{z}) = \bar{u}_L(\bar{x},\bar{z}) + \frac{\bar{u}^2(\bar{x},\bar{z})}{2} - \frac{1}{2\pi}\int\!\!\int_{-\infty}^{+\infty} \frac{\bar{u}^2(\bar{\xi},\bar{\zeta})}{2} \frac{[(\bar{x}-\bar{\xi})^2-(\bar{z}-\bar{\zeta})^2]}{[(\bar{x}-\bar{\xi})^2+(\bar{z}-\bar{\zeta})^2]^2} d\bar{\xi}d\bar{\zeta} \tag{104}$$

Approximate solutions of this equation could conceivably be worked out numerically by starting with a two-dimensional grid of suitably selected values of $\bar{u}$ and iterating until convergence is obtained. If the first approximation for $\bar{u}$ is taken to be the results given by incompressible or linearized subsonic flow theory, convergence will be obtained only when M_∞ is sufficiently small and the flow is subsonic everywhere. Oswatitsch suggested that mixed flow fields containing shock waves can be obtained if the starting $\bar{u}$-distribution contains shock waves. The idea then is to start with a reasonable guess for $\bar{u}$, being sure to include a proper discontinuity complying with the shock relations of Equation (95), then proceed to the solution. It is not necessary to be highly accurate in the initial guess for $\bar{u}$.

A further simplifying assumption is made by Oswatitsch and others regarding the variation of $\bar{u}$ in the z-direction, which has the effect of reducing the double integral in Equation (104) to a single integral. Oswatitsch suggested the following relation:

$$\bar{u}(\bar{x},\bar{z}) = \frac{\bar{u}_B(\bar{x},0)}{[1 + \frac{\bar{z}}{b}]^2} \tag{105}$$

where $\bar{u}_B(\bar{x},0)$ is the value of $\bar{u}$ on the body as taken in the linearized theory and b is a function of $\bar{x}$ so chosen that the irrotationality condition $\frac{\partial\bar{u}}{\partial\bar{z}} = \frac{\partial\bar{w}}{\partial\bar{x}}$ is fulfilled at $\bar{z} = 0$.

SECTION V

THE INTEGRAL EQUATION FOR THE THREE-DIMENSIONAL LIFTING WING-BODY COMBINATION

We start with the nonlinear partial differential Equation (61):

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = k\ \phi_x \phi_{xx} \tag{106}$$

where

$$k = M_\infty^2\ \frac{\nu + 1}{U_\infty} \tag{107}$$

The change of type of the equation is an essential feature of transonic flow and the regions where this occurs are recognized by the sign of the coefficient of ϕ_{xx}:

$$\left[1 - M_\infty^2 - M_\infty^2(\nu + 1)\ \frac{\phi_x}{U_\infty}\right] \begin{array}{ll} > 0 & \text{elliptic (subsonic)} \\ = 0 & \text{parabolic (sonic)} \\ < 0 & \text{hyperbolic (supersonic)} \end{array} \tag{108}$$

Equation (106) is, of course, valid only in regions where the necessary derivatives exist and are continuous. Across a shock surface the normal component of velocity is discontinuous while the tangential component, and therefore ϕ, is continuous. The necessary relation follows from the classical expression for the shock polar given in Section IV. This approximates, in small disturbance transonic theory, to [see Equation (68)]:

$$(1 - M_\infty^2)(\phi_{x_1} - \phi_{x_2})^2 + (\phi_{y_1} - \phi_{y_2})^2 + (\phi_{z_1} - \phi_{z_2})^2$$

$$= k\ \frac{\phi_{x_1} + \phi_{x_2}}{2}\ (\phi_{x_1} - \phi_{x_2})^2 \tag{109}$$

where the subscripts 1 and 2 refer to conditions ahead of and behind the shock.

Equations (106) and (109) are usually applicable to the case where the x-axis is parallel to the free-stream at infinity and are good approximations where the coordinate system is rotated slightly. Choosing the body-axis system, where x-axis is aligned in the longitudinal axis of the wing and body, the relation between the total velocity potential $\Phi(x,y,z)$ and the perturbation velocity potential $\phi(x,y,z)$ is approximated by:

$$\Phi(x,y,z) = U_\infty(x + \alpha z) + \phi(x,y,z) \tag{110}$$

where α is the angle of attack.

The expression for the pressure coefficient C_p is not invariant with respect to small rotations of the coordinate system. In body axes, the proper expression is

$$C_p = -\frac{2}{U_\infty}(\phi_x + \alpha\phi_z) - \frac{1}{U_\infty^2}(\phi_y^2 + \phi_z^2) \tag{111}$$

The Boundary Conditions

The condition at infinity yields

$$\Phi(\infty) = U_\infty(x + \alpha z)$$

or that

$$\phi(\infty) = 0 \quad . \tag{112}$$

An exception to this statement occurs in the vicinity of the wake at great distances behind the wing, but no complications arise due to the relative smallness of this region. The conditions at the airplane surface result in the relation

$$\frac{\partial\Phi}{\partial\tilde{n}} \equiv U_\infty(n_1 + \alpha n_3) + n_1 \frac{\partial\phi}{\partial x} + n_2 \frac{\partial\phi}{\partial y} + n_3 \frac{\partial\phi}{\partial z} = 0 \quad (113)$$

where n_1, n_2 and w_3 are the direction cosines of a normal to the airplane surface with respect to the x, y, and z axes, respectively. This relation is too general. For our needs $n_1 << 1$ or $(n_2^2 + n_3^2)^{1/2}$, i.e., slender bodies and wings are considered. Then Equation (113) reduces to

$$U_\infty(n_1 + \alpha n_3) + n_2 \frac{\partial\phi}{\partial y} + n_3 \frac{\partial\phi}{\partial z} = 0 \quad (114)$$

or

$$U_\infty(n_1 + \alpha n_3) + \frac{\partial\phi}{\partial n} = 0$$

where n is the normal to the curve bounding a cross section in a plane normal to the x-axis.

We will now apply Green's theorem in two different forms, one for $M_\infty \leq 1$, and the other for $M \geq 1$. Let

$$\beta = (|1 - M_\infty{}^2|)^{1/2} \quad ; \quad k = M_\infty \frac{\gamma + 1}{U_\infty} \quad (115)$$

Case 1. Subsonic Freestream Flow: $M_\infty \leq 1$

Equation (106) can be rewritten as

$$\beta^2\phi_{xx} + \phi_{yy} + \phi_{zz} = k\phi_x\phi_{xx} = k \frac{\partial}{\partial x} \left(\frac{\partial_x{}^2}{2}\right) \quad (116)$$

and the corresponding expression of Green's theorem is

$$\iiint_V [\psi L(\Omega) - \Omega L(\psi)] dV = - \iint_\Sigma \left(\psi \frac{\partial\Omega}{\partial\nu} - \Omega \frac{\partial\psi}{\partial\nu}\right) d\Sigma \quad (117)$$

where Ω and ψ are arbitrary functions and L is the Prandtl-Glauert operator:

$$L \equiv \beta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{118}$$

and $\frac{\partial}{\partial \nu}$ is the derivation along the conormal defined by

$$\frac{\partial}{\partial \nu} = \beta^2 n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z} \tag{119}$$

where n_1, n_2, n_3 are the direction cosines of the normal to the surface drawn into the region V.

Now choose $\psi = \frac{1}{\sigma}$, where

$$\sigma = \left[(x - \xi)^2 + \beta^2\{(y - \eta)^2 + (z - \zeta)^2\}\right]^{1/2} \tag{120}$$

so that $L(\psi) = 0$ and let $\Omega \equiv \phi$, the perturbation velocity potential, in Equation (117). Then from Equations (116) and (117) we obtain

$$\iint_\Sigma \left[\frac{1}{\sigma} \frac{\partial \phi}{\partial \nu} - \phi \frac{\partial}{\partial \nu} \left(\frac{1}{\sigma}\right)\right] d\Sigma = - \iiint_V \frac{k}{\sigma} \frac{\partial}{\partial \xi} \left(\frac{\phi_\xi^2}{2}\right) dV \tag{121}$$

In these equations the running coordinates in the integrations are ξ, η, ζ and ϕ is to be calculated at point P(x,y,z).

If Equation (121) is applied to the infinite region V consisting of a sphere of large radius which forms the external boundary V, a sphere of infinitesimal radius surrounding the field point P(x,y,z), and a final surface enveloping the wing-body, its wake, and its shock waves, we obtain the following expression:

$$\phi(x,y,z) = -\frac{1}{4\pi}\iint_{B+W}\left[\frac{1}{\sigma}\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial}{\partial\nu}\left(\frac{1}{\sigma}\right)\right]d\Sigma - \frac{1}{4\pi}\iint_{\lambda_1}\left[\frac{1}{\sigma}\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial}{\partial\nu}\left(\frac{1}{\sigma}\right)\right]d\Sigma$$

$$-\frac{1}{4\pi}\iint_{\lambda_2}\left[\frac{1}{\sigma}\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial}{\partial\nu}\left(\frac{1}{\sigma}\right)\right]d\Sigma - \frac{k}{4\pi}\iiint_V \frac{1}{\sigma}\frac{\partial}{\partial\xi}\left(\frac{\phi_\xi^{\,2}}{2}\right)dV \quad . \tag{122}$$

where B+W denotes the wing-body surface and its wake and λ_1 and λ_2 are the shocks, upstream and downstream shock surfaces, respectively. On the shock surface λ the conormals are directly opposed on the upstream and downstream faces λ_1 and λ_2. On the body itself and the wake, as mentioned earlier, we may approximate

$$\frac{\partial}{\partial\nu} \simeq n_2\frac{\partial}{\partial y} + n_3\frac{\partial}{\partial z} = \frac{\partial}{\partial n} \quad .$$

If now the triple integral in Equation (122) is integrated by parts twice with respect to x, the resultant form of Equation (122) is

$$\phi(x,y,z) = -\frac{1}{4\pi}\iint_{B+W}\left[\frac{1}{\sigma}\left(\frac{\partial\phi}{\partial n} - \frac{k}{2}\phi_\xi^{\,2}n_1\right) - \phi\frac{\partial}{\partial n}\frac{1}{\sigma}\right]d\Sigma$$

$$-\frac{1}{4\pi}\iint_{\lambda_1}\left[\frac{1}{\sigma}\left(\frac{\partial\phi}{\partial n} - \frac{k}{2}\phi_\xi^{\,2}n_1\right) - \phi\frac{\partial}{\partial n}\frac{1}{\sigma}\right]d\Sigma - \frac{1}{4\pi}\iint_{\lambda_2}\left[\frac{1}{\sigma}\left(\frac{\partial\phi}{\partial n} - \frac{k}{2}\phi_\xi^{\,2}n_1\right) - \phi\frac{\partial}{\partial n}\frac{1}{\sigma}\right]d\Sigma$$

$$+\frac{k}{4\pi}\iiint_V \frac{1}{2}\phi_\xi^{\,2}\left(\frac{\partial}{\partial\xi}\frac{1}{\sigma}\right)dV \quad . \tag{123}$$

Equation (123) is of particular interest because the integrals over the shock surfaces may be shown to vanish as in the nonlifting case. Finally, if we neglect the term $n_1\phi_\xi^{\,2}$ in comparison with $\partial\phi/\partial\nu$, the simplified integral equation is similar to the nonlifting case:

$$\phi(x,y,z) = -\frac{1}{4\pi}\iint_{B+W}\left(\frac{1}{\sigma}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\frac{1}{\sigma}\right)d\Sigma + \frac{k}{4\pi}\iiint_V \frac{1}{2}\phi_\xi{}^2\left(\frac{\partial}{\partial\xi}\frac{1}{\sigma}\right)dV \tag{124}$$

The first integral on the right-hand side of Equation (124) is the expression for $\phi(x,y,z)$ in linearized theory and the spatial integral is a contribution brought about by the nonlinear term of the basic differential Equation (106). It is of interest to remark that a straightforward derivation ignoring the existence of the shock waves leads to the same equation as Equation (124). For the majority of cases of practical interest, it can be shown that compensating terms arise and the discontinuity surfaces are taken care of by a formal development that ignores the existence of these surfaces. However, if it is not possible to ignore so completely the existence of the discontinuity surface, Equation (122) will be preferable to Equation (124).

Since

$$\frac{\partial}{\partial\xi}\frac{1}{\sigma} = -\frac{\partial}{\partial x}\frac{1}{\sigma} \quad ,$$

Equation (124) may be written in the final forms as

$$\phi(x,y,z) = \phi_L(x,y,z) + \frac{k}{4\pi}\iiint_V \frac{1}{2}\phi_\xi{}^2\left(\frac{\partial}{\partial\xi}\frac{1}{\sigma}\right)dV \tag{125}$$

or

$$\phi(x,y,z) = \phi_L(x,y,z) - \frac{k}{4\pi}\frac{\partial}{\partial x}\iiint \frac{1}{2}\phi_\xi{}^2\frac{1}{\sigma}dV \ . \tag{126}$$

Introducing the notation

$$\Delta\left(\frac{\partial\phi}{\partial\nu}\right) = \left(\frac{\partial\phi}{\partial\nu}\right)_{\lambda_1} + \left(\frac{\partial\phi}{\partial\nu}\right)_{\lambda_2}$$

and the variable

$$\omega = \sinh^{-1} \frac{x - \varepsilon}{\beta[(y-\eta)^2 + (z-\zeta)^2]^{1/2}}$$

$$\equiv \frac{(x-\xi)}{|x-\xi|} \ln \frac{|x-\xi| + \{(x-\xi)^2 + \beta^2(y-\eta)^2 + \beta^2(z-\zeta)^2}{\beta[(y-\eta)^2 + (z-\zeta)^2]^{1/2}}$$

and remembering that ϕ is continuous at the shock surface the more exact Equation (122) may be written in the following two alternative forms:

$$\phi(x,y,z) = \phi_L(x,y,z) - \frac{1}{4\pi} \iint_\lambda \Delta\left(\frac{\partial\phi}{\partial\nu}\right) \frac{1}{\sigma} d\Sigma - \frac{k}{4\pi} \iiint_V \left(\frac{\partial}{\partial\xi} \frac{\phi_\xi^2}{2}\right) \frac{1}{\sigma} dV \tag{127}$$

or

$$\phi(x,y,z) = \phi_L(x,y,z) - \frac{1}{4\pi} \frac{\partial}{\partial x} \iint_\lambda \Delta\left(\frac{\partial\phi}{\partial\nu}\right) \omega d\Sigma - \frac{k}{4\pi} \frac{\partial}{\partial x} \iiint_V \left(\frac{\partial}{\partial\xi} \frac{\phi_\xi^2}{2}\right) \omega dV \tag{128}$$

The perturbation velocity components, u, v and w, are obtained by differentiating ϕ with respect to x,y,z, respectively from either Equation (125/126) or Equation (127/128).

Consider Equation (125) for the x-derivative. After isolating the singularity at the field point by introducing the limits $\xi = x \pm \varepsilon$ and using the rule of differentiating under the sign of integration with variable limits:

$$\frac{d}{ds}\int_{g_1(s)}^{g_2(s)} F(x;s)\,dx = \int_{g_1(s)}^{g_2(s)} \frac{\partial F}{\partial s}\,dx + F\{g_1(s);s\}\frac{dg_1}{ds}$$

$$- F\{g_2(s);s\}\frac{dg_2}{ds}$$

we have

$$u(x,y,z) = u_L(x,y,z) + \lim_{\varepsilon\to 0}\frac{k}{4\pi}\frac{\partial}{\partial x}\iint d\eta d\zeta\left[\int_{-\infty}^{x-\varepsilon}\frac{1}{2}\phi_\xi{}^2\left(\frac{\partial}{\partial\xi}\frac{1}{\sigma}\right)d\xi\right.$$

$$\left.+\int_{x+\varepsilon}^{\infty}\frac{1}{2}\phi_\xi{}^2\left(\frac{\partial}{\partial\xi}\frac{1}{\sigma}\right)d\xi\right]$$

$$= u_L(x,y,z) + \lim_{\varepsilon\to 0}\frac{k}{2\pi}\iint\frac{1}{2}\phi_\xi{}^2(x;\eta,\zeta)\frac{\varepsilon\, d\eta d\zeta}{[\varepsilon^2+\beta^2(y-\eta)^2+\beta^2(z-\zeta)^2]^{3/2}}$$

$$-\frac{k}{4\pi}\iiint_V\frac{1}{2}\phi_\xi{}^2\left(\frac{\partial^2}{\partial\xi^2}\frac{1}{\sigma}\right)dV$$

In the limit as $\varepsilon\to 0$ the influence function in the integrand of the double integral is a two-dimensional Dirac delta function at the point $\eta = y$, $\zeta = z$ and of strength $2\pi/\beta^2$. The expression for u then becomes

$$u(x,y,z) = u_L(x,y,z) + \frac{k}{\beta^2}\frac{u^2(x,y,z)}{2} - \frac{k}{4\pi}\iiint_V\frac{u^2}{2}\left(\frac{\partial^2}{\partial\xi^2}\frac{1}{\sigma}\right)dV \tag{129}$$

Again starting with Equation (125) and differentiating with respect to y, we have

$$\frac{\partial\phi(x,y,z)}{\partial y} = \frac{\partial}{\partial y}\phi_L(x,y,z) + \frac{k}{4\pi}\lim_{\varepsilon\to 0}\frac{\partial}{\partial y}\left[\iint d\xi d\zeta \int_{-\infty}^{y-\varepsilon} \frac{1}{2}\phi_\xi{}^2\left(\frac{\partial}{\partial\xi}\frac{1}{\sigma}\right)d\eta\right.$$

$$\left. + \int_{y+\varepsilon}^{+\infty} \frac{1}{2}\phi_\xi{}^2\left(\frac{\partial}{\partial\xi}\frac{1}{\sigma}\right)d\eta\right]$$

or

$$v(x,y,z) = v_L(x,y,z) - \frac{k}{4\pi}\iiint_V \frac{1}{2}\phi_\xi{}^2\left(\frac{\partial^2}{\partial\xi\partial\eta}\frac{1}{\sigma}\right)dV$$

$$+ \lim_{\varepsilon\to 0}\left[\frac{k}{4\pi}\iint \frac{1}{2}\phi_\xi{}^2(\xi,y;\zeta)d\xi d\zeta\left\{\frac{x-\xi}{[(x-\xi)^2+\beta^2\varepsilon^2+\beta^2(z-\zeta)^2]^{3/2}}\right.\right.$$

$$\left.\left. - \frac{x-\xi}{[(x-\xi)^2+\beta^2\varepsilon^2+\beta^2(z-\zeta)^2]^{3/2}}\right\}\right]$$

or finally

$$v(x,y,z) = v_L(x,y,z) - \frac{k}{4\pi}\iiint_V \frac{u^2}{2}\left(\frac{\partial^2}{\partial\xi\partial\eta}\frac{1}{\sigma}\right)dV \qquad (130)$$

where we have made use of the fact that

$$\frac{\partial}{\partial y}\frac{1}{\sigma} = -\frac{\partial}{\partial\eta}\frac{1}{\sigma}$$

and the double integral tends to zero in the limit for all values of ξ and ζ .

Similarly it is easily seen, by differentiating Equation (125) with respect to z, that

$$w(x,y,z) = w_L(x,y,z) - \frac{k}{4\pi} \iiint_V \frac{u^2}{2} \left(\frac{\partial^2}{\partial\xi\partial\zeta} \frac{1}{\sigma} \right) dV \tag{131}$$

<u>Case 2. Integral Equation for M $\geq$ 1</u>

If the undisturbed flow at infinity is supersonic, i.e., $M_\infty \geq 1$, Equation (106) can be written as

$$- \beta^2 \phi_{xx} + \phi_{yy} + \phi_{zz} = k\, \phi_x \phi_{xx} = k \frac{\partial}{\partial x} \left(\frac{\phi_x^{\ 2}}{2} \right) \tag{132}$$

and the corresponding expression of Green's theorem is

$$\iiint_V \left[\psi \bar{L}(\Omega) - \Omega \bar{L}(\psi) \right] dV = - \iint_\Sigma \left(\psi \frac{\partial \Omega}{\partial \bar{\nu}} - \Omega \frac{\partial \psi}{\partial \bar{\nu}} \right) d\Sigma \tag{133}$$

where

$$\bar{L} \equiv - \beta^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{134}$$

and

$$\frac{\partial}{\partial \bar{\nu}} \equiv - \beta^2 n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z} \quad . \tag{135}$$

The derivation of the integral equation in Case 1 would suggest that ψ be replaced by $[(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2(z-\zeta)^2]^{-1/2}$ but this leads to the introduction of a finite-part technique in the integration. For the initial stages of the analysis, ψ will be identified with the fundamental solution

$$\bar{\omega} = \cosh^{-1} \frac{x - \xi}{\beta[(y-\eta)^2 + (z-\zeta)^2]^{1/2}}$$

of $\bar{L}(\psi) = 0$ used by Volterra (see Reference 14, page). Then from Equations (132) and (133), we have

$$\iint_{\bar{\Sigma}} \left(\bar{\omega} \frac{\partial \phi}{\partial \bar{\nu}} - \phi \frac{\partial \bar{\omega}}{\partial \bar{\nu}}\right) d\Sigma = - \iiint_{\bar{V}} \bar{\omega}\, \bar{L}(\phi) d\bar{V}$$

$$= - \iiint_{\bar{V}} k\, \bar{\omega} \frac{\partial}{\partial \xi} \left(\frac{\phi_\xi^2}{2}\right) d\bar{V} \qquad (136)$$

We have now to choose properly the three-dimensional region $\bar{V}$ and its enclosing surface $\bar{\Sigma}$. Discontinuities in the velocity components are again to be taken into consideration at the shock waves. Furthermore, the fundamental solution of Volterra becomes infinite at $\eta = y$, $\zeta = z$, that is, everywhere along the line passing through the field point P and parallel to the x-axis. Figure 11 indicates the disturbance field of the body as well as $\bar{V}$ and $\bar{\Sigma}$ (dashed lines). The bow shock fixes the foremost extent of the disturbance field and $\bar{\Sigma}$ lies adjacent to it and other possible shock surfaces as well as the surface of the body and its wake. The downstream limits of region $\bar{V}$ are fixed by the forecone with vortex P and determined explicitly by the relation

$$x - \xi = \beta[(y-\eta)^2 + (z-\zeta)^2]^{1/2} \qquad (137)$$

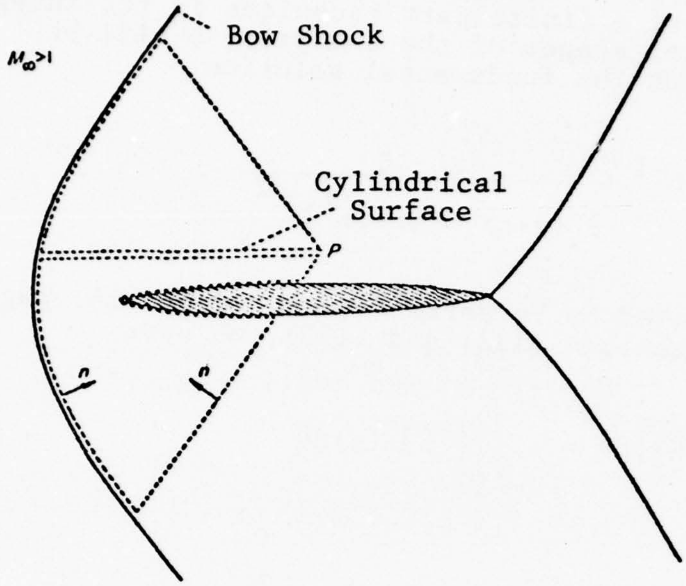


Figure 11. Region of Integration for $M_\infty \geq 1$.

The inner boundary of $\overline{V}$ is the cylindrical surface of infinitesimal radius given by the relation

$$(y-\eta)^2 + (z-\zeta)^2 = \varepsilon^2 \quad .$$

The conormal derivative is defined by Equation (135). On the infinitesimal cylinder its direction is parallel to that of the normal to the surface, and on the forecone from P the conormal is directed along the surface itself. We, therefore, obtain from Equation (136) the following expression for $\phi(x,y,z)$:

$$\phi(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial x}\iint_{\tau}\left(\bar{\omega}\frac{\partial\phi}{\partial\bar{\nu}} - \phi\frac{\partial\bar{\omega}}{\partial\bar{\nu}}\right)d\bar{\Sigma}$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial x}\iint_{\lambda_1}\left(\bar{\omega}\frac{\partial\phi}{\partial\bar{\nu}} - \phi\frac{\partial\bar{\omega}}{\partial\bar{\nu}}\right)d\bar{\Sigma} - \frac{1}{2\pi}\frac{\partial}{\partial x}\iint_{\lambda_2}\left(\bar{\omega}\frac{\partial\phi}{\partial\bar{\nu}} - \phi\frac{\partial\bar{\omega}}{\partial\bar{\nu}}\right)d\bar{\Sigma}$$

$$-\frac{k}{2\pi}\frac{\partial}{\partial x}\iiint \bar{\omega}\frac{\partial}{\partial\xi}\left(\frac{\phi_\xi^{\,2}}{2}\right)d\bar{V} \quad , \tag{138}$$

where integrals over the surface of the body and wake are denoted by τ, over the two sides of the shock surfaces by λ_1 and λ_2, and over the enclosed volume by $\bar{V}$. In each case, only that portion of the surface or volume lying within the forecone of P is included in the integrals. The surface integrals over the forecone itself vanish because $\bar{\omega}$ and $\partial\bar{\omega}/\partial\bar{\nu}$ are zero on it (Reference 14).

Integration by parts, in the volume integral in Equation (138), leads to the relation:

$$\phi(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial x}\iint_{\tau}\left[\bar{\omega}\left(\frac{\partial\phi}{\partial n} - \frac{k}{2}\phi_\xi^{\,2}n_1\right) - \phi\frac{\partial\bar{\omega}}{\partial n}\right]d\bar{\Sigma}$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial x}\left(\iint_{\lambda_1} + \iint_{\lambda_2}\right)\left[\bar{\omega}\left(\frac{\partial\phi}{\partial\bar{\nu}} - \frac{k}{2}\phi_\xi^{\,2}n_1\right) - \phi\frac{\partial\bar{\omega}}{\partial\bar{\nu}}\right]d\bar{\Sigma}$$

$$+\frac{k}{2}\frac{\partial}{\partial x}\iiint_{V}\frac{1}{2}\phi_\xi^{\,2}\left(\frac{\partial\bar{\omega}}{\partial\xi}\right)d\bar{V} \quad . \tag{139}$$

Equation (139) is the form for $M_\infty \gtrsim 1$ analogous to Equation (123) for $M_\infty \lesssim 1$ and on the body and wake surface involves the approximation

$$\frac{\partial}{\partial\bar{\nu}} \simeq n_2\frac{\partial}{\partial y} + n_3\frac{\partial}{\partial z} \simeq \frac{\partial}{\partial n}$$

As before, it can be shown from the shock-wave relation that the combined integrals over the surfaces λ_1 and λ_2 vanish. Therefore

$$\phi(x,y,z) = -\frac{1}{2\pi}\frac{\partial}{\partial x}\iint_{\tau}\left(\bar{\omega}\frac{\partial\phi}{\partial n} - \phi\frac{\partial\bar{\omega}}{\partial n}\right)d\bar{\Sigma}$$

$$+\frac{k}{2\pi}\frac{\partial}{\partial x}\iiint_{\bar{V}}\frac{1}{2}\phi_\xi^{\,2}\left(\frac{\partial\bar{\omega}}{\partial\xi}\right)d\bar{V} \quad . \tag{140}$$

Since

$$\bar{\omega} = \cosh^{-1}\frac{x-\xi}{\beta[(y-\eta)^2+(z-\zeta)^2]^{1/2}}$$

$$\frac{\partial\bar{\omega}}{\partial\xi} = -\frac{1}{[(x-\xi)^2-\beta^2(y-\eta)^2-\beta^2(z-\zeta)^2]^{1/2}} \equiv -\frac{1}{\sigma} \tag{141}$$

Equation (140) may be written in the following two alternative forms:

$$\phi(x,y,z) = \phi_L(x,y,z) + \frac{k}{2\pi}\frac{\partial}{\partial x}\iiint_{\bar{V}}\frac{1}{2}\phi_\xi^{\,2}\left(\frac{\partial\bar{\omega}}{\partial\xi}\right)d\bar{V} \tag{142}$$

or

$$\phi(x,y,z) = \phi_L(x,y,z) - \frac{k}{2\pi}\frac{\partial}{\partial x}\iiint_{\bar{V}}\frac{1}{2}\phi_\xi^{\,2}\frac{1}{\sigma}\,d\bar{V} \tag{143}$$

Similarly, Equation (138) may be given in the following two alternative forms using the notation

$$\Delta\frac{\partial\phi}{\partial\bar{\nu}} \equiv \left(\frac{\partial\phi}{\partial\bar{\nu}}\right)_{\lambda_1} + \left(\frac{\partial\phi}{\partial\bar{\nu}}\right)_{\lambda_2}$$

$$\phi(x,y,z) = \phi_L(x,y,z) - \frac{1}{2\pi}\frac{\partial}{\partial x} \iint_{\lambda} \Delta \left(\frac{\partial\phi}{\partial\bar{\nu}}\right) \bar{\omega} d\bar{\Sigma}$$

$$- \frac{k}{2\pi}\frac{\partial}{\partial x} \iiint_{\bar{V}} \left(\frac{\partial}{\partial\xi}\frac{\phi_\xi^2}{2}\right)\bar{\omega} d\bar{V} \tag{144}$$

or

$$\phi(x,y,z) = \phi_L(x,y,z) - \frac{1}{2\pi} \iint_{\lambda} \Delta \left(\frac{\partial\phi}{\partial\bar{\nu}}\right) \frac{1}{\sigma} d\bar{\Sigma}$$

$$- \frac{k}{2\pi} \iiint_{\bar{V}} \left(\frac{\partial}{\partial\xi}\frac{\phi_\xi^2}{2}\right) \frac{1}{\sigma} d\bar{V} \tag{145}$$

where ϕ_L is the value from the linearized supersonic theory.

Conditions at the Shock Surface

It is of some interest to study Equation (125) or Equation (126) as the field point approaches a discontinuity surface and to discover the mechanism by means of which these basic equations furnish the velocity jumps associated with the shock waves in the field. The surface of the wave can be replaced locally by a planar element and a new coordinate system X,Y,Z introduced with the origin fixed at the intersection of the line $\eta = y$, $\zeta = z$ and the shock surface. Then the point P, at which conditions are to be calculated close to the surface of the wave. has the coordinates (X,0,0) and the planar surface of the wave is given by the linear relation

$$aX_1 + bY_1 + cZ_1 = 0 \; . \tag{146}$$

We assume that u^2 in Equation (125) is composed of a continuous part and a discontinuous part that has the constant value u_1^2 ahead of and u_2^2 behind the shock. Equation (125) then yields

$$\lim_{X \to 0} [u(X_-,0,0) - u(X_+,0,0)]$$

$$= \lim_{X \to 0} \frac{k}{4\pi} (u_1^2 - u_2^2) \frac{\partial}{\partial X} \iint \frac{a\, dY_1\, dZ_1}{[(aX+bY_1+cZ_1)^2 + a^2\beta^2 {Y_1}^2 + a^2\beta^2 {Z_1}^2]^{1/2}} \tag{147}$$

where the double integral extends over the region of the discontinuity. If the differentiation with respect to X is now carried out within the integral signs and X allowed to approach zero, we will obtain

$$u_1 - u_2 = \lim_{X \to 0} [u(X_-,0,0) - u(X_+,0,0)] = \frac{ka^2}{a\beta^2 + b^2 + c^2} \frac{u_1^2 - u_2^2}{2} \tag{148}$$

It can easily be shown that Equation (148) agrees with the result given by Equation (68), the shock polar condition, for $M_\infty \leq 1$.

SECTION VI

SOLUTION OF THE TRANSONIC THREE-DIMENSIONAL INTEGRAL EQUATION FOR THE NONLIFTING CASE

We have described the difficulties and complications involved in the solution of two-dimensional transonic nonlinear equation. As a consequence of the obstacles encountered in the two-dimensional case, attempts at the solution of the nonlinear equation with three space variables are very few. In fact, only two references can be cited which can be regarded as a proper three-dimensional analysis. The first one is the work by Alksne and Spreiter (Reference 15) which is actually an extension of the local linearization method previously developed for planar (Reference 16) and axisymmetric flow (Reference 17). This has been summarized below. The second work is by Norstrud who, for the first time, attempts to solve the nonlinear integral equation by analytical-cum-numerical technique. This needs detailed review and discussion.

1. THE ALKSNE-SPREITER METHOD OF SOLUTION OF THE TRANSONIC THREE-DIMENSIONAL EQUATION FOR NONLIFTING WINGS

Essentially this method is an extension of the local linearization technique so successfully employed by the authors for two-dimensional thin airfoils in transonic flows (Reference 16).

The axis system, as shown in Figure 12, consists of the Cartesian coordinates with x-axis parallel to the free-stream and with the origin at the west forward point of the wing. The wing sections are given by $z = Z(x,y)$ and the wing planform is described by $y = -S_1(x)$ and $y = S_2(x)$ as shown. For all practical wings we will take $|S_{2max}| = |S_{1max}| = b/2$, the semispan.

If ϕ is the perturbation velocity potential, whose gradient yields the perturbation velocity components u, v and w, parallel to the x, y and z axes, the problem of transonic flow around thin finite wings can be studied by using the usual nonlinear equation.

$$(1 - M_\infty^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = \frac{M_\infty^2(\gamma+1)}{U_\infty} \phi_x\phi_{xx} \equiv k\,\phi_x\phi_{xx} \tag{149}$$

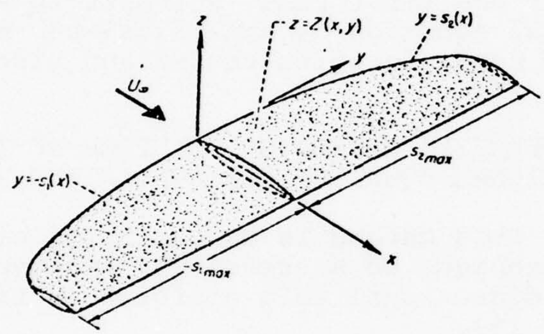


Figure 12. The Wing and the Coordinate System.

The appropriate boundary conditions are:

(1) $u, v, w \to 0$ far ahead of the wing, and

(2) the flow be tangential to the wing surface.

The latter condition is satisfied if

$$(\phi_z)_{z=0} = U_\infty \frac{\partial Z(x,y)}{\partial x} \quad . \tag{150}$$

In addition to the above condition it is necessary to take proper account of the difference in regions of influence and dependence in the subsonic and supersonic portion of the flow field. In general, additional equations are needed for flows across the shock. Alksne and Spreiter did not consider them for the following reasons:

(1) In the first place, attention was restricted to M_∞ very close to 1 so that the flow in the vicinity of the wing is essentially the same, by virtue of the Mach number freeze, as at $M_\infty \simeq 1$.

(2) Secondly, they assumed that the shock stands at the rear of the wing where its influence cannot extend onto the wing surface.

Approximate Solution

First, the coefficient of ϕ_x is replaced by the symbol λ as follows:

$$\lambda = \frac{M_\infty^2(\gamma+1)}{U_\infty} \phi_{xx} \equiv k \frac{\partial u}{\partial x} > 0 \tag{151}$$

so that Equation (149) becomes:

$$\phi_{yy} + \phi_{zz} - \lambda \phi_x = -(1 - M_\infty^2)\phi_{xx} \quad . \tag{152}$$

They have restricted their attention to only accelerated flows for which $\lambda > 0$. In the initial stages of the analysis it is assumed that λ is a constant.

The resulting Equation (152) is linear for λ = constant and is of elliptic, hyperbolic or parabolic type depending on whether $M \lesseqgtr 1$. Use of the boundary conditions given above and the application of Green's theorem to Equation (149) leads to the following equation for u if shock conditions are not taken into account:

$$u_P(x,y,z) = -\frac{U_\infty}{2\pi}\frac{\partial}{\partial x}\int_0^x d\xi \int_{-S_1(\xi)}^{S_2(\xi)} \frac{\partial Z/\partial\xi}{x-\xi}\, e^{-\lambda\frac{[(y-\eta)^2+z^2]}{4(x-\xi)}}\, d\eta$$

$$-\frac{1}{\lambda}\frac{\partial}{\partial\xi}\int_{-\infty}^{+\infty} d\eta \int_{-\infty}^{+\infty} d\zeta \int_{-\infty}^{x} \frac{f_P}{x-\xi}\, e^{-\lambda\frac{[(y-\eta)^2+z^2]}{4(x-\xi)}}\, d\xi \tag{153}$$

where

$$f_P = -\,(1 - M_\infty^2)\,\phi_{\xi\xi}\ .$$

At $M_\infty = 1$, fp = 0 and up can be calculated for any given value of λ for a known profile and planform. At other M_∞ Equation (153) is an integral equation whose solution remains to be found. For M_∞ near 1, however, f_P is very small and $\phi_{\xi\xi}$ can safely be replaced by λ/k from Equation (151). The triple integral can then be evaluated giving us the following result:

$$\frac{u_P(x,y,z)}{U_\infty} = \frac{1-M_\infty^2}{M_\infty^2(\gamma+1)} - \frac{1}{2\pi}\frac{\partial}{\partial x}\int_0^x d\xi \int_{-S_1(\xi)}^{S_2(\xi)} \frac{\partial Z/\partial\xi}{x-\xi}\, e^{-\lambda\frac{[(y-\eta)^2+z^2]}{4(x-\xi)}}\, d\eta \tag{154}$$

Next, the value of $k\phi_{xx}$ at the point x,y,z is restored in place of λ in Equation (154). Then for each value of y and z the result is a first-order, nonlinear, ordinary differential equation for u as a function of x. Dropping the subscript P and writing $k\phi_{xx}$ as k u', we obtain from Equation (154):

$$\frac{u(x,y,z)}{U_\infty} =$$

$$\left[\frac{1-M_\infty^2}{M_\infty^2(\gamma+1)} - \frac{1}{2\pi}\frac{\partial}{\partial x}\right]_{u'=\text{const}} \int_0^x d\xi \int_{-S_1(\xi)}^{S_2(\xi)} \frac{\partial Z/\partial \xi}{x-\xi}\, e^{\frac{-ku'[(y-\eta)^2+z^2]}{4(x-\xi)}}\, d\eta \tag{155}$$

It is convenient to express Equation (155) in terms of reduced variables indicated by the transonic similarity rule and defined as follows:

$$\bar{x} = \frac{x}{c}\ , \quad \frac{dZ}{dx} = \tau\,\frac{d(Z/t)}{d\bar{x}}\ , \quad \tau = \left(\frac{t}{c}\right)_{max}\ ,$$

$$\bar{y} = [M_\infty^2(\gamma+1)\tau]^{1/3}\,\frac{y}{c}\ , \quad \bar{z} = [M_\infty^2(\gamma+1)\tau]^{1/3}\,\frac{z}{c}\ ,$$

$$\bar{s} = [M_\infty^2(\gamma+1)\tau]^{1/3}\,\frac{s}{c}\ , \quad \bar{\eta} = [M_\infty^2(\gamma+1)\tau]^{1/3}\,\frac{\eta}{c}\ ,$$

$$\bar{\xi} = \frac{\xi}{c}\ , \quad \xi_\infty = \frac{M_\infty^2-1}{[M^2(\gamma+1)\tau]^{2/3}}\ ,$$

$$\bar{u} = \frac{[M_\infty^2(\gamma+1)]^{1/3}}{\tau^{2/3}}\,\frac{u}{U_\infty} = -\frac{\bar{C}_p}{2} \tag{156}$$

Equation (155) thus becomes

$$\bar{u}(\bar{x},\bar{y},\bar{z}) =$$

$$\left[-\xi_\infty - \frac{1}{2\pi}\frac{\partial}{\partial \bar{x}}\right]_{\bar{u}'=\text{const}} \int_0^{\bar{x}} d\bar{\xi} \int_{-\bar{S}_1(\bar{\xi})}^{\bar{S}_2(\bar{\xi})} \frac{\partial(Z/t)/\partial\bar{\xi}}{\bar{x}-\bar{\xi}}\, e^{\frac{-\bar{u}'[(\bar{y}-\bar{\eta})^2+\bar{z}^2]}{4(\bar{x}-\bar{\xi})}}\, d\bar{\eta} \tag{157}$$

In cases where interest is confined to conditions on the wing surface, $\bar{u}$ can be evaluated with $\bar{z} = 0$ and Equation (157) can be simplified to become

$$\bar{u}(\bar{x},\bar{y},0) =$$

$$-\xi_0 - \frac{1}{2\pi}\frac{\partial}{\partial\bar{x}}\Bigg]_{\bar{u}'=\mathrm{const}} \int_0^{\bar{x}} d\bar{\xi} \int_{-\bar{S}_1(\xi)}^{\bar{S}_2(\xi)} \frac{\partial(Z/t)/\partial\bar{\xi}}{\bar{x}-\bar{\xi}}\, e^{\frac{-\bar{u}'(\bar{y}-\bar{\eta})^2}{4(\bar{x}-\bar{\xi})}}\, d\bar{\xi} \tag{158}$$

Properties of Transonic Flow

From Equation (158) it can be seen that for affinely related planforms (fixed $\bar{s}(\bar{x})$) and affinely related profiles (fixed $\partial(Z/t)/\partial\bar{x}$), $\bar{u}$ at a given $\bar{x}$ and $\bar{y}$ varies only with

$$\xi_\infty = \frac{M_\infty^2 - 1}{[M_\infty^2(\gamma+1)\tau]^{2/3}} .$$

Thus the present results follow the transonic similarity rule. Note also that $\bar{s}$ has the same parametric form as the aspect ratio parameter customarily used for the transonic similarity rules, i.e., $\bar{A} = M_\infty^2(\gamma+1)\tau]^{1/3}$ A, where A is the aspect ratio.

Furthermore, $\bar{u}'$ in the exponent in Equation (158) can be replaced by $\frac{d(\bar{u}+\xi_\infty)}{d\bar{x}}$ for a given value of $\bar{y}$ and it follows for wings of affinely related geometry that $\bar{u}+\xi_\infty$ at a given $\bar{x}$ and $\bar{y}$ does not vary with ξ_∞. Thus the results given by Alksne and Spreiter contain the Mach number freeze. Notice also that there is a coupling of aspect ratio, thickness ratio, and Mach number in $\bar{A}$, such that in order to maintain a fixed $\bar{A}$ for a wing of given thickness ratio as M_∞ varies it is necessary to vary the aspect ratio likewise. The variation of aspect ratio with M_∞ is not great in the range for which the freeze can be expected to apply.

Solution of Equation (157)

For each value of $\bar{y}$ and $\bar{z}$ Equation (157) is a first order ordinary differential equation and its solution requires the specification of an initial or a boundary condition. The simplest case is one for which a value of $\bar{u}$ is known at some value of $\bar{x}$ for every spanwise station and z = 0. This is the case for a wing of wedge profile, for which the sonic velocity occurs at the shoulder. In general, however, there is no point at which $\bar{u}$ is known *a priori* and some alternative condition must be imposed in order to determine a unique solution. It happens in all cases considered, that is, two-dimensional flow around an airfoil, axisymmetric flow around a body of revolution, and now three-dimensional flow past a wing, that the equation contains a singular point through which pass an infinity of integral curves. In each case only one is analytic. Thus the requirement of analyticity is sufficient to determine a unique solution.

Mathematically, this leads to the following condition: Writing Equation (158) as

$$\bar{U} = F(\bar{x},\bar{U}';\bar{y}) \tag{159}$$

where

$$\bar{U} = \bar{u} + \xi_\infty$$

and

$$\bar{U}' = \bar{u}' = \frac{d(\bar{u}+\xi_\infty)}{d\bar{x}}$$

the equation will be investigated for a fixed value of $\bar{y}$. From Equation (159), we have

$$\bar{U}' = \frac{\partial F}{\partial \bar{x}} + \bar{U}'' \frac{\partial F}{\partial \bar{u}'} \tag{160}$$

where, in general, $\partial F/\partial \bar{x}$ and $\partial F/\partial \bar{U}'$ are functions of $\bar{x}$ and $\bar{U}'$.

At a point where $\partial F/\partial \bar{U}' = 0$ we can find $\bar{U}'$ provided that $\bar{U}''$ is finite, but this is assured if the solution is analytic. With $\bar{U}'$ known at a point, it is possible to calculate $\bar{U}$ at that point using Equation (159). This is not enough to determine a unique solution because $\bar{U}''$ can

have any finite value. However, additional derivatives can be taken and in each case the unknown, but finite, higher derivative of $\overline{U}$ will be multiplied by $\partial F/\partial \overline{U}'$ which is equated to zero. Thus all derivations can be determined in principle at the singular point.

The location of the singular point and the velocity gradient at that point are then obtained by solving simultaneously the two equations:

$$\frac{\partial F}{\partial \overline{U}'} = 0 \quad \text{and} \quad \overline{U}' = \frac{\partial F}{\partial \overline{x}} \quad .$$

Special Cases

Alksne and Spreiter have considered several special cases and deduced closed form expressions for Equation (157) or Equation (158).

(1) Central Region of High Aspect Ratio Wing

By letting $\overline{S}_1(\overline{x}) = \overline{S}_2(\overline{x}) = \infty$ and considering Z a function of $\overline{x}$ only across the entire span, the following asymptotic form of Equation (158) results:

$$(\overline{u} + \xi_\infty) \quad \overline{u}' = - \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^{\overline{x}} \frac{d(Z/t)/d\xi}{(\overline{x} - \xi)^{1/2}} d\xi \tag{161}$$

If the sonic point, $\overline{x}^*$, where $\overline{u} + \xi_\infty = 0$, can be found or is known, the general solution may be written as

$$\begin{aligned} -\overline{u} &= \xi_\infty - \left\{ \frac{3}{\pi} \int_{\overline{x}^*}^{\overline{x}} \left[\frac{d}{d\overline{x}_1} \int_0^{\overline{x}_1} \frac{d(Z/t)/d\xi}{(\overline{x} - \xi)^{1/2}} d\xi \right]^2 d\overline{x}_1 \right\}^{1/3} \\ &= \xi_\infty - [\overline{u}]_{\xi_\infty = 0} \end{aligned} \tag{162}$$

For a smooth profile for which $\bar{u} + \xi_\infty$ is not known *a priori* at any point, the singular point is found to be the value of $\bar{x}$ for which

$$\frac{d}{d\bar{x}} \int_0^{\bar{x}} \frac{d(Z/t)/d\bar{\xi}}{(\bar{x} - \bar{\xi})^{1/2}} d\xi = 0 \quad . \tag{163}$$

Using this value of $\bar{x}^*$ in Equation (162) a unique solution is determined.

Theoretical pressure distributions computed for several airfoils using the above equations are in good agreement with experimental pressure distributions at $M_\infty = 1$ (Reference 16).

(2) Tip of High Aspect-Ratio Wing

Placing the origin at the tip so that $\bar{y} = \bar{s}_1 = 0$ and $\bar{s}_2(\bar{x})$ can be set equal to infinity, Z can then be considered a function of $\bar{x}$ alone across the entire span. Equation (158) then becomes

$$\bar{u} + \xi_\infty = -\frac{1}{2}\left[\frac{1}{\pi \bar{u}'} \frac{d}{d\bar{x}} \int_0^{\bar{x}} \frac{d(Z/t)/d\bar{\xi}}{(\bar{x} - \bar{\xi})^{1/2}} d\bar{\xi}\right] \tag{164}$$

This is the same result as Equation (161) with a factor 1/2 in Equation (164) which cannot affect the position of the singular point. It follows therefore

$$\bar{C}_{p_{tip}} = 2\xi_\infty + \left(\frac{1}{2}\right)^{2/3} \left(\bar{C}_{p_{\xi_\infty = 0}}\right)_{2\text{-dim.}}$$

or (165)

$$\bar{u}_{tip} = \left(\frac{1}{2}\right)^{2/3} \left(\bar{u}_{\xi_\infty = 0}\right)_{2\text{-dim.}} - \xi_\infty$$

(3) Small Aspect-Ratio Wing

In Equation (155) let the integration be performed in two steps: 0 to $x - \varepsilon$ and then from $x - \varepsilon$ to x, where ε is a small quantity.

$$\frac{u(x,y,z)}{U_\infty} = -\frac{1}{2\pi}\frac{\partial}{\partial x}\Bigg]_{u'=\text{const}} \left\{ \int_0^{x-\varepsilon} d\xi \int_{-S_1(\xi)}^{S_2(\xi)} \frac{\partial Z/\partial \xi}{x-\xi}\, e^{\frac{-ku'[(y-\eta)^2+z^2]}{4(x-\xi)}}\, d\eta \right.$$

$$\left. + \int_{x-\varepsilon}^{x} d\xi \int_{-S_1(\xi)}^{S_2(\xi)} \frac{\partial Z/\partial \xi}{x-\xi}\, e^{\frac{-ku'[(y-\eta)^2+z^2]}{4(x-\xi)}}\, d\eta \right\}$$

$$+ \frac{1 - M_\infty^2}{M_\infty^2(\gamma+1)} \tag{166}$$

Since the point $\xi = x$ is excluded from the first term on the right-hand side, the exponential term for a small aspect ratio wing can be replaced by 1. In the second integral we replace $\partial Z/\partial \xi \simeq \partial Z/\partial x$ in the small interval $(x-\varepsilon, x)$ and obtain the simpler equation:

$$\frac{u}{U_\infty} = -\frac{1}{2\pi}\frac{\partial}{\partial x}\Bigg]_{u'=\text{const}} \left\{ \int_0^{x-\varepsilon} \frac{d\xi}{x-\xi} \int_{-S_1(\xi)}^{S_2(\xi)} \frac{\partial Z}{\partial \xi}\, d\eta \right.$$

$$\left. + \int_{-S_1(x)}^{S_2(x)} \frac{\partial Z}{\partial x}\, d\eta \int_{x-\varepsilon}^{x} e^{\frac{-ku'[(y-\eta)^2 + Z^2]}{4(x-\xi)}} \frac{d\xi}{x-\xi} \right\}$$

$$+ \frac{1 - M_\infty^2}{M_\infty^2(\gamma + 1)} \tag{167}$$

Now

$$\int_{-S_1(\xi)}^{S_2(\xi)} \frac{\partial Z}{\partial \xi}\, d\eta = S'(\xi)$$

where S(x) is the cross-sectional area of the wing perpendicular to the x-direction if Z = 0 at the leading and trailing edges at any spanwise *station* and the ξ-integration of the second term on the right can be performed approximately by retaining only the first logarithmic term.

The desired asymptotic form for a wing of vanishing aspect-ratio becomes

$$\frac{u}{U_\infty} - \frac{1 - M_\infty^{\,2}}{M_\infty^{\,2}(\gamma+1)} = -\frac{S'(0)}{4\pi x} + \frac{S''(x)}{4\pi}\,\ell n\,\frac{M_\infty^{\,2}(\gamma+1)e^{C}}{4x}$$
$$+ \frac{S''(x)}{4\pi}\,\ell n\left(\frac{u'}{U_\infty}\right) + \frac{1}{4\pi}\int_0^x \frac{S''(x) - S''(\xi)}{x - \xi}\,d\xi$$
$$+ \frac{1}{2\pi}\frac{\partial}{\partial x}\int_{-S_1(x)}^{S_2(x)} \frac{\partial Z}{\partial x}\,\ell n\,[(y-\eta)^2 + z^2]\,d\eta \qquad (168)$$

where the Euler's constant C = 0.5772156 . . .

Remarks on the Alksne-Spreiter Method

The method is elegant and simple to use with fair degree of accuracy but it is very limited in scope as regards the flow conditions and the wing geometry. The following are some of the restrictions:

(a) The wings must have sharp leading edges with zero lift. Randall (Reference 18) has given an extension applicable, under certain circumstances, to round nosed *two-dimensional* airfoils at angles of attack.

(b) The Mach number should be very near one and the wing planform and profile should be such that at all points the flow accelerates along the chord.

SECTION VII

TRANSONIC THREE-DIMENSIONAL FLOW OVER ARBITRARY WING PLANFORMS: THE INTEGRAL EQUATION METHOD OF SOLUTION OF NORSTRUD

NONLIFTING CASE

The three-dimensional transonic integral equation for the nonlifting case has been derived in Section IV. We reproduce it here for convenience:

$$\bar{u}(\bar{x},\bar{y},\bar{z}) = \bar{u}_L(\bar{x},\bar{y},\bar{z}) + \frac{1}{2}\bar{u}^2(\bar{x},\bar{y},\bar{z}) - \frac{1}{4\pi}\iiint_{-\infty}^{+\infty} \frac{\bar{u}^2}{2}(\bar{\xi},\bar{\eta},\bar{\zeta}) \frac{\partial^2}{\partial\bar{\xi}^2}\left(\frac{1}{r_3}\right) d\bar{\xi}\, d\bar{\eta}\, d\bar{\zeta} \tag{169}$$

where $\bar{u}_L(\bar{x},\bar{y},\bar{z})$ is the result from the linearized theory and

$$\frac{\partial^2}{\partial\bar{\xi}^2}\left(\frac{1}{r_3}\right) = -\frac{1}{[(\bar{x}-\bar{\xi})^2+(\bar{y}-\bar{\eta})^2+(\bar{z}-\bar{\zeta})^2]^{3/2}}\left\{1-\frac{3(\bar{x}-\bar{\xi})^2}{(\bar{x}-\bar{\xi})^2+(\bar{y}-\bar{\eta})^2+(\bar{z}-\bar{\zeta})^2}\right\} \tag{170}$$

The physical interpretation of Equation (169) is that of a mutual interference between velocity points in a perturbed flow-field $u(x,y,z)$ where the linearized field $u_L(x,y,z)$ designates the datum for zero perturbation. The kernel of the triple integral serves as the influence measure and it can be seen from Equation (169) that this function is largely dependent on the double derivative in the $\bar{x}$-direction of the inverse of the distance r_3 between influencing points. This means that the strongest influence at a point $P(x,y,z)$ in the flow field will come from field points which are located directly upstream or downstream of P. The spanwise effects, however, are of major concern in describing spatial transonic flow. Here the compressibility of the flowing gas tends to freeze the available stream tube area and the density flux $\rho\bar{u}$ must be balanced with neighboring stream tubes. Norstrud assumes the following exponential functional relationship to extract the spanwise influence from Equation (169):

$$\bar{u}(\bar{x},\bar{y},\bar{z}) = \bar{u}(\bar{x},\bar{y},0)e^{-Z/r} \tag{171}$$

which gives a relation between the velocity in the $\bar{x}\bar{y}$-plane of the reduced wing and the velocities directly above. The realation [Equation (171)] satisfies correctly the boundary condition of zero $\bar{u}$-disturbance at infinity ($\bar{z} \to \infty$). The parameter $r = r(\bar{x},\bar{y})$ is defined using the irrotationality condition and the linearized value $\bar{u}_L(\bar{x},\bar{y},0)$ over the wing which automatically satisfies the tangential flow condition:

$$r(\bar{x},\bar{y}) = - \frac{\bar{u}_L(\bar{x},\bar{y},0)}{\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)_{\bar{x},\bar{y},0}} \tag{172}$$

Using the irrotationality condition

$$\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)_{wing} = \left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)_{wing}$$

we have from the linearized boundary condition

$$r(\bar{x},\bar{y}) = - \frac{\bar{u}_L(\bar{x},\bar{y},0)}{\frac{\partial^2 \bar{Z}}{\partial \bar{x}^2}} \tag{173}$$

where $\bar{Z}$ represents the reduced ordinates of the wing section at station $\bar{y}$.

Using the tables of the Laplace transform, the Equation (169) can be rewritten, for $\bar{x} - \bar{\xi} = 0$, as

$$\bar{u}(\bar{x},\bar{y},\bar{z}) = \frac{1}{2}\bar{u}^2(\bar{x},\bar{y},\bar{z}) + \bar{u}_L(\bar{x},\bar{y},\bar{z}) - e^{-2\frac{Z}{r}} \iint_{-\infty}^{+\infty} \bar{u}^2(\bar{\xi},\bar{\eta},0)E_3(|\bar{y}-\bar{\eta}|;r)\,d\bar{\xi}\,d\bar{\eta} \tag{174}$$

The spanwise influence function $E_3 = E_3(|\bar{y}-\bar{\eta}|;r)$ is found, for the argument $\sigma = 2|\bar{y}-\bar{\eta}|/r$ (from Reference 19) to be

$$E_3(\sigma) = -\frac{2\pi}{\sigma}\left[\frac{2}{\pi} - H_1(\sigma) + N_1(\sigma)\right] \tag{175}$$

where H_1 is the Struve function of order one and N_1, the Neumann function also identified as the Bessel function of the second kind Y_1. $N_1(\sigma)$ can be calculated using a standard available computer subroutine. The series expansion for the numerical evaluation of the Struve function is

$$H_1(\sigma) = \left(\frac{\sigma}{2}\right)^2 \sum_{n=0}^{\infty} A_n \quad ,$$

where

$$A_0 = \frac{8}{3\pi} \quad , \quad A_{n+1} = -\frac{(\sigma/2)^2}{(n+3/2)(n+5/2)} A_n \quad .$$

In its limiting form for $|\sigma| \rightarrow 0$, Equation (175) is given by

$$E_3(\sigma) \simeq 4\left[\frac{1}{\sigma^2} - \frac{1}{\sigma} + \frac{1}{3}\sigma - \frac{1}{45}\sigma^3 + \ldots\right] \tag{176}$$

Using the exponential decay and applying a similar procedure for two-dimensional flows the influence function $E_2(X) = E_2(2\frac{|\bar{x}-\xi|}{r})$ in the x-direction is found to be

$$E_2(X) = \frac{4}{\pi}\{\sin X[\frac{\pi}{2} - \mathrm{Si}X] - \cos X \ \mathrm{Ci}\ X\}$$

where

$$\mathrm{Si}X = \int_0^X \frac{\sin t}{t}\,dt \quad \text{and} \quad \mathrm{Ci}X = -\int_X^\infty \frac{\cos t}{t}\,dt$$

For small values of the argument, we have

$$\mathrm{Si}X = \frac{X}{1.1!} - \frac{X^3}{3.3!} + \frac{X^5}{5.5!} - \ldots .$$

$$\mathrm{Ci}X = -C - \ell nX + \frac{X^2}{2.2!} - \frac{X^4}{4.4!} + \ldots .$$

where

$$C = \text{Euler's constant} \simeq 0.57721 \ldots .$$

and hence yields a logarithmic singularity for $|X| \to 0$. The rapid decrease in spanwise influence is typical in spatial flows. The lateral influence function $E_3(\sigma)$ is shown plotted in Figure 13.

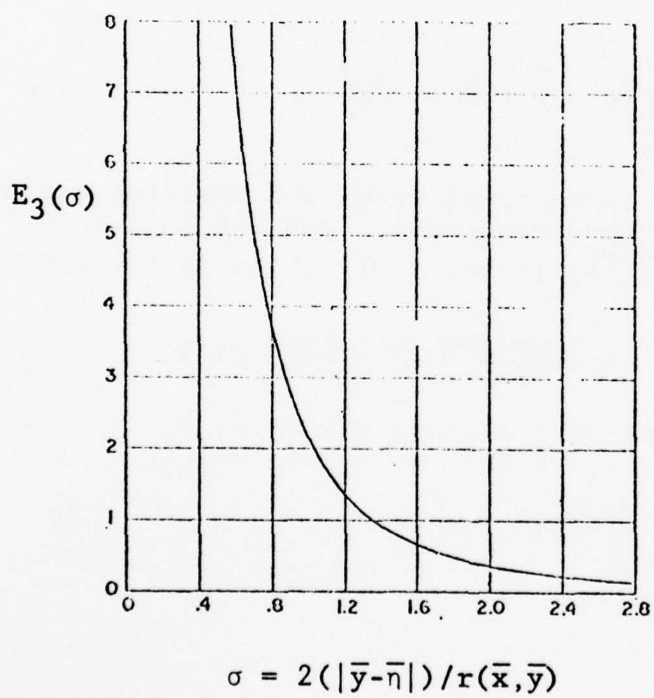


Figure 13. Lateral Influence Function $E_3(\sigma)$.

Norstrud's Approximate Method of Solution of Equation (169)

The symmetrical wing of arbitrary planform is represented by a number of rectangular panels in the xy-plane. The distribution used for these panels is shown in Figure 14.

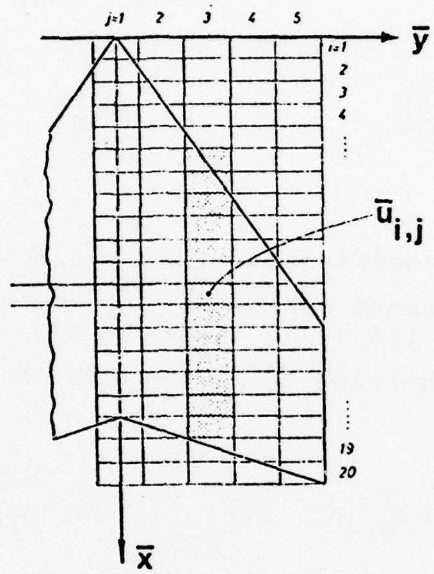


Figure 14. Approximate Representation of the Wing.

The fundamental point of view adopted by Norstrud in the definition of the influence region for a projected point of interest $(x_o, y_o, 0)$ on the wing for the evaluation of the double integral in Equation (174) is to collect the effects of all field points located at a common streamwise station $(x = x_o, x_1, \ldots, x_n)$ and assume this total influence to be represented at the point (x, y_o, o) by a mean value obtained from two-dimensional analysis. These influences which may constitute the solution from a strip method are supplemented by the effects of laterally located points, i.e., for $x = x_o$ and $y \neq y_o$. The influence pattern is illustrated in Figure 14 by the shaded area and the resulting influence matrix. [I] has the general structure

$$I_{i,j}^{k,\ell} \equiv \begin{vmatrix} (\frac{1}{2} - \varepsilon_{1,1}^{1,1}) & \varepsilon_{1,1}^{2,1} & \varepsilon_{1,1}^{k,\ell} & \varepsilon_{1,1}^{n,m} \\ \varepsilon_{2,1}^{1,1} & (\frac{1}{2} - \varepsilon_{2,1}^{2,1}) & \varepsilon_{2,1}^{k,\ell} & \varepsilon_{2,1}^{n,m} \\ \cdots & \cdots & \cdots & \\ \varepsilon_{i,j}^{1,1} & \varepsilon_{i,j}^{2,1} & (\frac{1}{2} - \varepsilon_{i,j}^{k,\ell}) & \varepsilon_{i,j}^{n,m} \\ \cdots & \cdots & \cdots & \\ \varepsilon_{n,m}^{1,1} & \varepsilon_{n,m}^{2,1} & \varepsilon_{n,m}^{k,\ell} & (\frac{1}{2} - \varepsilon_{n,m}^{n,m}) \end{vmatrix}$$

The influence coefficients $\varepsilon_{i,j}^{k,\ell}$ are integral functions of the defined influence functions E_2 and E_3 for the case where $\ell = j$ and $k = i (\ell \neq j)$, respectively. If $\ell \neq j$ and $k \neq i$, $\varepsilon_{i,j}^{k,\ell} = 0$. Equation (174) can now be written in the matrix form

$$[\bar{u}_{ij}] + [I_{i,j}^{k,\ell}][\bar{u}_{k,\ell}^{2}] - [\bar{u}_{L_{i,j}}] = 0, \quad \begin{matrix} i,k = 1,2, \ldots, n \\ j,\ell = 1,2, \ldots, m \end{matrix} \tag{177}$$

and the problem has been reduced to that of solving a system of nonlinear algebraic equations. Norstrud used the Newton-Raphson method of successive approximations to solve the system [Equation (177)]. This utilizes a linear system at each iteration step.

REFERENCES

1. Liepmann, H. W., and Roshko, A. "Elements of Gas-dynamics"; John Wiley and Sons, N.Y., 1967.

2. Lomax, H., Bailey, F. R., and Ballhaus, W. F. "On the Numerical Simulation of Three-Dimensional Transonic Flow with Application to C-141 Wing"; NASA TN D-6933, 1973.

3. Bagley, J. A. "Some Aerodynamic Principles for the Design of Swept Wings"; Progress in Aero. Sci., Vol. 3, Ed. Ferri, Juchemann and Sterne, Pergamon Press, N.Y., 1962.

4. Lax, P. D. "Weak Solutions of Nonlinear Hyperbolic Equations and Their Numerical Computation"; Comm. Pure and App. Maths, Vol. 7, Feb. 1965, pp. 159-193.

5. Steger, J. L., and Baldwin, B. S. "Shock Waves and Drag in the Numerical Calculation of Isentropic Transonic Flow"; NASA TN D-6997, 1972.

6. Klunker, E. B., and Newman, P. A. "Computation of Transonic Flow About Lifting Wing-Cylinder Combinations"; J. Aircraft, Vol. 11, No. 4, April 1974.

7. Hall, M. G., and Firmin, M.C.P. "Recent Developments in Methods for Calculating Transonic Flows Over Wings"; Int. Council of Aero. Sci., Congress, 9th, Haifa, Israel, Aug. 1974, ICAS paper 74-18.

8. Albone, C. M., Hall, M. G., and Joyce, G. "Numerical Solutions for Transonic Flows Past Wing-Body Combinations"; Symposium Transsonicum II, Ed. K. Oswatitsch and D. Rues, Gottingen, 1975, Springer-Verlag, Berlin.

9. Van der Vooren, J., Sloff, J. W., Huizing, G. H., and van Essen, A. "Remarks on the Suitability of Various Transonic Small Perturbation Equations to Describe Three-Dimensional Transonic Flow; Examples of Computations Using a Fully Conservative Rotated Difference Scheme"; Symposium Transsonicum II, Ed. K. Oswatitsch, Gottingen, 1975.

10. Spreiter, J. R., and Alksne, A. Theoretical Prediction of Pressure Distributions on Nonlifting Airfoils at High Subsonic Speeds, NACA Report 1217, 1955.

11. Nixon, D. Extended Integral Equation Method for Transonic Flows, AIAA Journal, Tech. Notes, Vol. 13, No. 7, 1975, pp. 934-935.

12. Nixon, D. Calculation of Transonic Flows Using an Extended Integral Equation Method, AIAA Journal, Vol. 15, No. 3, 1977.

13. Heaslet, M. A., and Spreiter, J. R. Three-Dimensional Transonic Flow Theory Applied to Slender Wings and Bodies, NACA Rep. 1318, 1957.

14. Sears, W. R., Editor: General Theory of High Speed Aerodynamics, Vol. VI, Princeton Univ. Press, 1954.

15. Alksne, A. Y., and Spreiter, J. R. Theoretical Pressure Distributions on Wings of Finite Span at Zero Incidence for Mach Numbers Near 1, NASA Tech. Rep. R-88, 1961.

16. Spreiter, J. R., and Alksne, A. Y. Thin Airfoil Theory Based on an Approximate Solution of the Transonic Flow Equation. NACA TN 3970, 1957.

17. Spreiter, J. R., and Alksne, A. Y. Slender-Body Theory Based on Approximate Solution of the Transonic Flow Equation. NASA Tech. Rep. R-2, 1959.

18. Randall, D. G. Transonic Flow Over Two-Dimensional Round-Nosed Aerofoils"; British A.R.C. Tech. Rep., C.P. No. 456, 1959.

19. Roberts, G. E., and Kaufman, H. Table of Laplace Transform. W. B. Saunders Co., Philadelphia, 1966.

20. Norstrud, H. High Speed Flow Past Wings, NASA CR-2246, 1973.

21. Yakovlev, M. N. The Solution of Systems of Nonlinear Equations by a Method of Differentiation with Respect to a Parameter. U.S.S.R. Computational Mathematics and Mathematical Physics, Vol. 4, No. 1, pp. 198-203.

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